An Equal-Space Construction of Integrable Models

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Talk Outline

- Defining terms
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- The equal-space picture
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  - Lax pairs
  - Example
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Defining Terms

Equal-time $\iff$ time evolution $\iff$ space-like.
Equal-space $\iff$ space evolution $\iff$ time-like.

Using the zero-curvature form of Lax pairs (U, V), so that the equations of motion are given by [Lax '68; Ablowitz, *et al.* '74]:

$$0 = \partial_t U - \partial_x V + [U, V],$$

and the Lax matrices obey the auxiliary linear problem:

$$\partial_x \Psi = U\Psi, \qquad \partial_t \Psi = V\Psi.$$

The equal-time Poisson brackets for an integrable system given in terms of a Lax pair can be written in terms of a classical r-matrix [Sklyanin '79]:

$$\{U_1(x;\lambda), U_2(y;\mu)\}_S = [r_{12}(\lambda-\mu), U_1(x;\lambda) + U_2(y;\mu)]\delta(x-y),$$

where r_{12} satisfies the classical Yang-Baxter equation [Semenov-Tian-Shansky '83]:

$$[r_{12}(\lambda-\mu), r_{13}(\lambda)] + [r_{12}(\lambda-\mu), r_{23}(\mu)] + [r_{13}(\lambda), r_{23}(\mu)] = 0.$$

The time evolution of the system is then defined through the Poisson bracket and the Hamiltonian:

$$\partial_t \mathfrak{f} = \{H_S, \mathfrak{f}\}_S.$$

We consider two types of boundary conditions for x in the interval $x \in [-L, L]$:

- Periodic: $\mathfrak{f}(L) = \mathfrak{f}(-L)$ for all fields $\mathfrak{f}$.
- Reflective: We need to introduce some K-matrices to sit at the boundaries. They must to satisfy [Sklyanin '87]:

$$0 = K_{\pm,1}(\lambda) r_{12}(\lambda + \mu) K_{\pm,2}(\mu) - K_{\pm,2}(\mu) r_{12}(\lambda + \mu) K_{\pm,1}(\lambda) + [r_{12}(\lambda - \mu), K_{\pm,1}(\lambda) K_{\pm,2}(\mu)].$$

We only consider non-dynamical boundary conditions, so that $\partial_t K_{\pm} = 0$.

To generate conserved quantities, define the monodromy matrix as a solution of the spatial part of the ALP in place of Ψ:

$$T_S(x,y;\lambda) = \mathsf{P}\ \mathsf{exp} \int_y^x U(\xi;\lambda)\mathsf{d}\xi.$$

For periodic boundary conditions, the trace of this is called the transfer matrix:

$$\mathfrak{t}_S(\lambda) = \mathsf{tr} T_S(L,-L;\lambda),$$

while for reflective boundary conditions, the transfer matrix is [Sklyanin '87]:

$$\bar{\mathfrak{t}}_S(\lambda) = \mathsf{tr}\big\{K_+(\lambda)T_S(L,-L;\lambda)K_-(\lambda)T_S^{-1}(L,-L;-\lambda)\big\}.$$

We want local quantities, so we consider $\mathcal{G}_S(\lambda) = \ln \mathfrak{t}_S(\lambda)$ and $\bar{\mathcal{G}}_S(\lambda) = \ln \bar{\mathfrak{t}}_S(\lambda)$. Expanding these about powers of λ:

$$\mathcal{G}_S(\lambda) = \sum_k \lambda^k \mathcal{G}_S^{(k)}, \qquad \bar{\mathcal{G}}_S(\lambda) = \sum_k \lambda^k \bar{\mathcal{G}}_S^{(k)},$$

we find local quantities that Poisson commute with one another:

$$\{\mathcal{G}_S^{(k)}, \mathcal{G}_S^{(j)}\}_S = 0 = \{\bar{\mathcal{G}}_S^{(k)}, \bar{\mathcal{G}}_S^{(j)}\}_S.$$

If we call one of these $\mathcal{G}_S^{(k)}$ or $\bar{\mathcal{G}}_S^{(k)}$ the Hamiltonian, then this provides us with a hierarchy of conserved Poisson commuting quantities.

Each Hamiltonian defines an integrable system. As they were derived from the U-matrix, the V-matrix will be what differs between them.

The V-matrices associated to each of the Hamiltonians (with periodic boundary conditions) are taken from the expansion of [Semenov-Tian-Shansky '83]:

$$\mathbb{V}_2(x;\lambda,\mu) = \mathfrak{t}_S^{-1}(\mu)\mathrm{tr}_1\big\{T_{S,1}(L,x;\mu)r_{12}(\mu-\lambda)T_{S,1}(x,-L;\lambda)\big\}.$$

For reflective boundary conditions, we instead expand (where we use $r^{\pm} = r(\mu \pm \lambda)$) [Avan, Doikou '08]:

$$\bar{\mathbb{V}}_2(x) = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_1\big\{K_{+,1}(\mu)T_{S,1}(L,x;\mu)r^-_{12}T_{S,1}(x,-L;\lambda)K_{-,1}(\mu)T^{-1}_{S,1}(-\lambda) + K_{+,1}(\mu)T_{S,1}(\mu)K_{-,1}(\mu)T^{-1}_{S,1}(x,-L;-\lambda)r^+_{12}T^{-1}_{S,1}(L,x;-\lambda)\big\}.$$

Consider the isotropic Landau-Lifshitz model:

$$\partial_t \vec{S} = \mathrm{i}\vec{S} \times (\partial_x^2 \vec{S}),$$

where $\vec{S} = (S_1, S_2, S_3)^T$ and $S_1^2 + S_2^2 + S_3^2 = |\vec{S}|^2 = 1$. The Lax pair for this model is [Lakshmanan '77; Takhtajan '77]:

$$U = \frac{1}{2\lambda}\begin{pmatrix} S_3 & S_1 - \mathrm{i}S_2 \\ S_1 + \mathrm{i}S_2 & -S_3 \end{pmatrix} \equiv \frac{1}{2\lambda}S, \qquad V = \frac{1}{2\lambda^2}S - \frac{1}{2\lambda}(\partial_x S)S.$$

The U-matrix combines with the r-matrix $r = \frac{1}{2\lambda}\mathcal{P}$, where $\mathcal{P}$ is the permutation matrix, to give the following Poisson brackets:

$$\{S_i(x), S_j(y)\}_S = -\mathrm{i}\epsilon_{ijk}S_k(x)\delta(x-y).$$

The first two non-trivial (periodic) conserved quantities that we generate from the U-matrix are:

$$H_S^{(0)} = \frac{\mathsf{i}}{2}\int_{-L}^{L} \frac{S_2\partial_x S_1 - S_1\partial_x S_2}{1+S_3}\mathsf{d}x, \qquad H_S^{(1)} = \frac{-1}{4}\int_{-L}^{L}\sum_i(\partial_x S_i)^2\mathsf{d}x,$$

and the associated V-matrices are (up to constant factors):

$$V^{(0)} = \frac{1}{2\lambda}S, \qquad V^{(1)} = \frac{1}{2\lambda^2}S - \frac{1}{2\lambda}(\partial_x S)S.$$

The conserved quantities can be recognised as the total momentum and Hamiltonian (up to constant factors), and the V-matrices can be recognised as the two components of the Lax pair.

We can construct an equivalent set of Poisson brackets which govern the "space evolution" of the fields [Caudrelier, Kundu '15; Avan, *et al.* '16].

For symmetry with the equal-time approach, we want to write these brackets in terms of an r-matrix:

$$\{V_1(t_1;\lambda), V_2(t_2;\mu)\}_T = [r_{12}(\lambda-\mu), V_1(t_1;\lambda) + V_2(t_2;\mu)]\delta(t_1-t_2).$$

The r-matrix we would use is the same as the r-matrix from the equal-time approach.

The "space evolution" of the system is then defined through this equal-space Poisson bracket and some spatially conserved Hamiltonian:

$$\partial_x \mathfrak{f} = \{H_T, \mathfrak{f}\}_T.$$

Like in the equal-time picture, we consider two types of boundary conditions for t in the interval $t \in [-\tau, \tau]$:

- Periodic: $\mathfrak{f}(\tau) = \mathfrak{f}(-\tau)$ for all fields $\mathfrak{f}$.
- Reflective: We again need some K-matrices to describe the boundaries, which are solutions of the classical reflection equation.

  As we use the same r-matrix, we can use the same K-matrices.

  We still only consider non-dynamical boundary conditions, although we mean spatially independent this time, i.e. $\partial_x K_\pm = 0$.

The equal-space monodromy matrix is built as a solution to the *time* component of the auxiliary linear problem, $\partial_t \Psi = V\Psi$:

$$T_T(t_1, t_2; \lambda) = \mathsf{P}\ \mathsf{exp} \int_{t_2}^{t_1} V(\xi; \lambda) \mathsf{d}\xi.$$

The equal-space transfer matrix is then found by taking the trace of this (for periodic boundary conditions):

$$\mathfrak{t}_T(\lambda) = \mathsf{tr} T_T(\tau, -\tau; \lambda),$$

and the reflective version is similarly analogous to the equal-time picture:

$$\bar{\mathfrak{t}}_T(\lambda) = \mathsf{tr}\big\{K_+(\lambda) T_T(\tau, -\tau; \lambda) K_-(\lambda) T_T^{-1}(\tau, -\tau; -\lambda)\big\}.$$

The (temporally) local quantities are found from the logarithm of these:

$$\mathcal{G}_T(\lambda) = \ln\, \mathfrak{t}_T(\lambda), \qquad \bar{\mathcal{G}}_T(\lambda) = \ln\, \bar{\mathfrak{t}}_T(\lambda),$$

which are expanded about powers of λ to give local Poisson commuting (with respect to the equal-space bracket) quantities:

$$\{\mathcal{G}_T^{(k)}, \mathcal{G}_T^{(j)}\}_T = 0 = \{\bar{\mathcal{G}}_T^{(k)}, \bar{\mathcal{G}}_T^{(j)}\}_T.$$

In analogy to relativistic models, where momentum is to space as energy to time, we can see that:

$$\text{If: } \left\{ \begin{matrix} \mathcal{G}_S^{(k)} = \text{Hamiltonian} \\ \mathcal{G}_S^{(j)} = \text{Momentum} \end{matrix} \right\} \text{ then: } \left\{ \begin{matrix} \mathcal{G}_T^{(k)} = \text{Dual Momentum} \\ \mathcal{G}_T^{(j)} = \text{Dual Hamiltonian} \end{matrix} \right\}$$

Just as we found the V-matrix from the U-matrix in the equal-time picture, we expect to be able to find the U-matrix from the V-matrix in the equal-space picture.

The generator for these U-matrices is given by:

$$\mathbb{U}_2(t;\lambda,\mu) = \mathfrak{t}_T^{-1}(\mu)\mathrm{tr}_1\big\{T_{T,1}(\tau,t;\mu)r_{12}(\mu-\lambda)T_{T,1}(t,-\tau;\lambda)\big\}.$$

For reflective boundary conditions, the generator is instead [Doikou, *et al.* '18]:

$$\begin{aligned}\bar{\mathbb{U}}_2(t) = \bar{\mathfrak{t}}_T^{-1}(\mu)\mathrm{tr}_1\big\{&K_{+,1}(\mu)T_{T,1}(\tau,t;\mu)r_{12}^{-}T_{T,1}(t,-\tau;\lambda)K_{-,1}(\mu)T_{T,1}^{-1}(-\lambda)\\ &+K_{+,1}(\mu)T_{T,1}(\mu)K_{-,1}(\mu)T_{T,1}^{-1}(t,-\tau;-\lambda)r_{12}^{+}T_{T,1}^{-1}(\tau,t;-\lambda)\big\}.\end{aligned}$$

We again use the isotropic Landau-Lifshitz equation as an example (now written in terms of its space evolution):

$$\partial_x \vec{S} = \vec{\Sigma}, \qquad \partial_x \vec{\Sigma} = \mathrm{i}\vec{S} \times (\partial_t \vec{S}) - \vec{S}|\vec{\Sigma}|^2,$$

where $\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)^T$ and $S_1\Sigma_1 + S_2\Sigma_2 + S_3\Sigma_3 = \vec{S} \cdot \vec{\Sigma} = 0$. Combining the V-matrix:

$$V = \frac{1}{2\lambda^2} S - \frac{1}{2\lambda} \begin{pmatrix} \Sigma_3 & \Sigma_1 - \mathrm{i}\Sigma_2 \\ \Sigma_1 + \mathrm{i}\Sigma_2 & -\Sigma_3 \end{pmatrix} S \equiv \frac{1}{2\lambda^2} S - \frac{1}{2\lambda} \Sigma S,$$

with the Yangian r-matrix from the equal-time picture $r = \frac{1}{2\lambda}\mathcal{P}$, we find the following equal-space Poisson brackets:

$$\begin{aligned}
&\{S_i(t_1), S_j(t_2)\}_T = 0, \\
&\{S_i(t_1), \Sigma_j(t_2)\}_T = \left(S_i S_j - \delta_{ij}\right) \delta(t_1 - t_2), \\
&\{\Sigma_i(t_1), \Sigma_j(t_2)\}_T = \left(S_i \Sigma_j - S_j \Sigma_i\right) \delta(t_1 - t_2).
\end{aligned}$$

The first two non-trivial (periodic) conserved quantities that we generate from the V-matrix are [Findlay '18]:

$$H_T^{(0)} = \frac{\mathsf{i}}{2}\int_{-\tau}^{\tau}\left[\frac{S_2\partial_t S_1 - S_1\partial_t S_2}{1+S_3} + \frac{\mathsf{i}}{2}(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2)\right]\mathrm{d}t,$$

$$\begin{aligned} H_T^{(1)} = \frac{-1}{2}\int_{-\tau}^{\tau}\Bigg[&\Sigma_3\partial_t S_3 + \partial_t\Sigma_3 + (\Sigma_1 - \mathsf{i}\Sigma_2)\partial_t(S_1+\mathsf{i}S_2) \\ &- S_3\Sigma_3\frac{\partial_t(S_1-\mathsf{i}S_2)}{S_1-\mathsf{i}S_2} + \frac{S_1+\mathsf{i}S_2}{1+S_3}\Bigg((\Sigma_1-\mathsf{i}\Sigma_2)\partial_t S_3 \\ &+\partial_t(\Sigma_1-\mathsf{i}\Sigma_2) - (\Sigma_1-\mathsf{i}\Sigma_2)\frac{\partial_t(S_1-\mathsf{i}S_2)}{S_1-\mathsf{i}S_2}\Bigg)\Bigg]\mathrm{d}t. \end{aligned}$$

The first of these turns out to be the equal-space Hamiltonian for this model, and the second does indeed (eventually) give a transport type equation:

$$\partial_x\vec{S} \propto \partial_t\vec{S}, \qquad \partial_x\vec{\Sigma} \propto \partial_t\vec{\Sigma}.$$

The first two non-trivial U-matrices are comparatively simple (after removing constant factors):

$$U^{(0)} = \frac{1}{2\lambda}S, \qquad U^{(1)} = \frac{1}{2\lambda^2}S - \frac{1}{2\lambda}\Sigma S,$$

which are just the U- and V-matrices in the Lax pair.

These match the first two V-matrices generated from the U-matrix, but at higher orders they diverge:

$$V^{(2)} = \frac{1}{2\lambda^3}S - \frac{1}{2\lambda^2}(\partial_x S)S + \frac{1}{2\lambda}\partial_x^2 S + \frac{3}{4\lambda}(\partial_x S)^2 S,$$
$$U^{(2)} = \frac{1}{2\lambda^3}S - \frac{1}{2\lambda^2}\Sigma S - \frac{1}{2\lambda}(\partial_t S)S + \frac{1}{4\lambda}\Sigma^2 S.$$

We consider the K-matrix [de Vega, González-Ruiz '94]:

$$K_{\pm} = a_{\pm}\mathbb{I} + \lambda \begin{pmatrix} d_{\pm} & \beta_{\pm} - \mathrm{i}\gamma_{\pm} \\ \beta_{\pm} + \mathrm{i}\gamma_{\pm} & -d_{\pm} \end{pmatrix}.$$

At $x = \pm L$ the space-like boundary conditions are [Doikou, Karaiskos '11]:

$$\begin{aligned} a_{\pm}(S_2\partial_x S_1 - S_1\partial_x S_2) &= \pm(\beta_{\pm}S_2 - \gamma_{\pm}S_1), \\ a_{\pm}(S_1\partial_x S_3 - S_3\partial_x S_1) &= \pm(d_{\pm}S_1 - \beta_{\pm}S_3), \\ a_{\pm}(S_2\partial_x S_3 - S_3\partial_x S_2) &= \pm(d_{\pm}S_2 - \gamma_{\pm}S_3). \end{aligned}$$

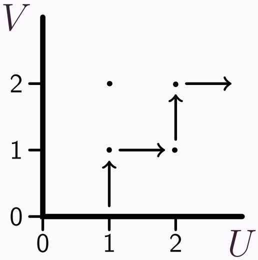
At $t = \pm\tau$ the time-like boundary conditions are [Findlay '18]:

$$a_{\pm} = 0, \qquad\qquad 0 = \beta_{\pm}S_1 + \gamma_{\pm}S_2 + d_{\pm}S_3.$$

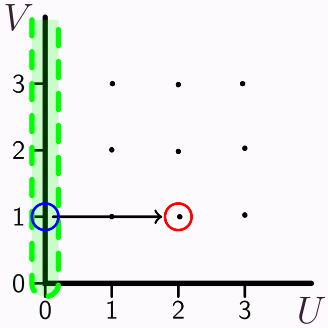
From Hierarchies to "Lattices"

In the equal-time picture, we get a hierarchy of time-flows.
In the equal-space picture, we get a hierarchy of space-flows.

By alternating which picture is considered, we can build a 2-dimensional "lattice" of integrable systems. This is not *a priori* commutative.



From Hierarchies to "Lattices"



Blue circle - The isotropic LL model

Green dashed region - The usual isotropic LL hierarchy

Red circle - A new, 6-field system

Start from the isotropic Landau-Lifshitz U-matrix. Then generate the V-matrix. Then use this to generate a *different* U-matrix:

$$U^{(0)} \to (U^{(0)}, V^{(1)}) \to (U^{(2)}, V^{(1)}).$$

This new model will have Lax pair:

$$U = \frac{1}{2\lambda^3}S - \frac{1}{2\lambda^2}\Sigma S - \frac{1}{2\lambda}(\partial_t S)S + \frac{1}{4\lambda}\Sigma^2 S, \qquad V = \frac{1}{2\lambda^2}S - \frac{1}{2\lambda}\Sigma S.$$

The corresponding equations of motion are [Findlay '18]:

$$\begin{aligned}
\partial_x \vec{S} &= \mathrm{i}\vec{S} \times (\partial_t \vec{\Sigma}) + \frac{1}{2}|\vec{\Sigma}|^2 \vec{\Sigma}, \\
\partial_x \vec{\Sigma} &= \mathrm{i}\vec{\Sigma} \times (\partial_t \vec{\Sigma}) - \frac{\mathrm{i}}{2}|\vec{\Sigma}|^2 \big(\vec{S} \times (\partial_t \vec{S})\big) + \mathrm{i}\vec{\Sigma}\big(\vec{\Sigma} \cdot (\vec{S} \times (\partial_t \vec{S}))\big) \\
&\quad + \partial_t^2 \vec{S} + \vec{S}(|\partial_t \vec{S}|^2 - \frac{1}{2}|\vec{\Sigma}|^4).
\end{aligned}$$

General Poisson Structure

Let's actually look at the *an*isotropic Landau-Lifshitz equation now. The equations of motion are [Landau, Lifshitz '65]:

$$\partial_t \vec{S} = \mathrm{i}\vec{S} \times (J\vec{S} + \partial_x^2 \vec{S}),$$

where $J = \mathrm{diag}(J_1, J_2, J_3)$, with $J_1 < J_2 < J_3$, describes the anisotropy.

The U component of the Lax pair is written [Sklyanin '79]:

$$U = \mathrm{i}\rho \begin{pmatrix} \mathrm{cs}(\lambda, k)S_3 & \mathrm{ns}(\lambda, k)S_1 - \mathrm{ids}(\lambda, k)S_2 \\ \mathrm{ns}(\lambda, k)S_1 + \mathrm{ids}(\lambda, k)S_2 & -\mathrm{cs}(\lambda, k)S_3 \end{pmatrix},$$

where:

$$\rho = \frac{1}{2}\sqrt{J_3 - J_1}, \qquad k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}.$$

The r-matrix associated to this Lax matrix is:

$$r(\lambda) = \frac{\mathrm{i}\eta\rho}{2}\left[\mathsf{cs}(\lambda,k)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} + \mathsf{ns}(\lambda,k)\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\right.$$
$$\left. +\mathsf{ds}(\lambda,k)\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\right].$$

This gives $\mathfrak{su}_2$-type Poisson brackets:

$$\{S_i(x), S_j(y)\}_S = -\mathrm{i}\epsilon_{ijk}S_k(x)\delta(x-y).$$

For any U-matrix compatible with this r-matrix, the associated V-matrices will all have the same form:

$$\mathbb{V} = \begin{pmatrix} \mathbb{S}_3(\mu)\mathsf{cs}(\mu-\lambda) & \mathbb{S}_1(\mu)\mathsf{ns}(\mu-\lambda) - \mathsf{i}\mathbb{S}_2(\mu)\mathsf{ds}(\mu-\lambda) \\ \mathbb{S}_1(\mu)\mathsf{ns}(\mu-\lambda) + \mathsf{i}\mathbb{S}_2(\mu)\mathsf{ds}(\mu-\lambda) & -\mathbb{S}_3(\mu)\mathsf{cs}(\mu-\lambda) \end{pmatrix},$$

where we are expanding in μ. Then, using the equal-space Poisson bracket expression, we can find the time-like Poisson structure between the coefficients in the of $\mathbb{S}_i(\mu)$ about powers of μ, labelled $\mathbb{S}_i^{(k)}$:

$$\{\mathbb{S}_i^{(p)}, \mathbb{S}_j^{(q)}\}_T = \begin{cases} \frac{p!q!}{(p+q-n)!}\mathbb{S}_k^{(p+q-n)}\epsilon_{ijk} & p+q \geq n, \\ 0 & p+q < n, \end{cases}$$

where n is the order in the hierarchy of the V-matrix that we are considering.

By defining:

$$S_i^{(n,p)} = \frac{1}{(n-p)!} \mathbb{S}_i^{(n-p)},$$

the time-like Poisson brackets between these fields are (with $0 \le p, q \le n$):

$$\{S_i^{(n,p)}(t_1), S_j^{(n,q)}(t_2)\}_T = \begin{cases} S_k^{(n,p+q)} \epsilon_{ijk}\, \delta(t_1 - t_2) & p+q \le n, \\ 0 & p+q > n. \end{cases}$$

An identical process gives the equal-time Poisson structure at order n as:

$$\{S_i^{(n,p)}(x), S_j^{(n,q)}(y)\}_S = \begin{cases} S_k^{(n,p+q)} \epsilon_{ijk}\, \delta(x - y) & p+q \le n, \\ 0 & p+q > n. \end{cases}$$

Let's look at some examples. In the equal-space hierarchy built out of the Landau-Lifshitz V-matrix, the Landau-Lifshitz U-matrix appears at order 0, so we choose $n = 0$ in the above formulae.

The formulae then tell us that the equal-time Poisson brackets for the Landau-Lifshitz U-matrix are:

$$\{S_i^{(0,0)}(x), S_j^{(0,0)}(y)\}_S = S_k^{(0,0)} \epsilon_{ijk}\, \delta(x-y),$$

which is just the usual $\mathfrak{su}_2$-type algebra.

The Landau-Lifshitz V-matrix appears at order 1 in the equal-time hierarchy generated by the U-matrix, so we choose $n = 1$ for that.

We then have two sets of fields, $S_i^{(1,0)}$ and $S_i^{(1,1)}$ which obey the equal-space Poisson brackets of the Landau-Lifshitz model:

$$\{S_i^{(1,0)}(t_1), S_j^{(1,0)}(t_2)\}_T = S_k^{(1,0)} \epsilon_{ijk}\, \delta(t_1 - t_2),$$
$$\{S_i^{(1,0)}(t_1), S_j^{(1,1)}(t_2)\}_T = S_k^{(1,1)} \epsilon_{ijk}\, \delta(t_1 - t_2),$$
$$\{S_i^{(1,1)}(t_1), S_j^{(1,1)}(t_2)\}_T = 0.$$

This is a special Euclidean SE(3)-type algebra, which is equivalent to the equal-space Poisson brackets given before, with:

$$S_i^{(1,1)} = S_i, \qquad S_i^{(1,0)} = (\vec{\Sigma} \times \vec{S})_i.$$

Non-Ultralocal Models

So far we have only discussed ultralocal models, where the Poisson brackets depended only on δ. For non-ultralocal models (where we also have a δ') the Poisson brackets are written in terms of an (r, s) pair instead of just the r-matrix [Maillet '86]:

$$\begin{aligned}\{U_1(x;\lambda), U_2(y;\mu)\}_S = [r_{12}(\lambda-\mu), U_1(x;\lambda) + U_2(y;\lambda)]\delta(x-y) \\ + [s_{12}(\lambda-\mu), U_1(x;\lambda) - U_2(y;\lambda)]\delta(x-y) \\ + 2s_{12}(\lambda-\mu)\delta'(x-y).\end{aligned}$$

The (r, s) pair needs to satisfy the general classical Yang-Baxter equation:

$$\begin{aligned}0 = [(r_{12}+s_{12})(\lambda-\mu), (r_{13}-s_{13})(\lambda)] \\ + [(r_{12}-s_{12})(\lambda-\mu), (r_{23}-s_{23})(\mu)] \\ + [(r_{13}-s_{13})(\lambda), (r_{23}-s23)(\mu)].\end{aligned}$$

We consider a simple example of a non-ultralocal model, the real modified Korteweg-de Vries equation:

$$\partial_t v = 6v^2 \partial_x v - \partial_x^3 v,$$

which has the following Poisson brackets and Hamiltonian:

$$\{v(x), v(y)\}_S = \delta'(x-y), \qquad H_S = \frac{1}{2}\int_{-L}^{L} (v\partial_x^2 v - v^4)\mathsf{d}x.$$

The Lax pair for this model can be written [Ablowitz, *et al.* '74]:

$$U = \begin{pmatrix} -\mathsf{e}^{\lambda} & \mathsf{i}v \\ -\mathsf{i}v & \mathsf{e}^{\lambda} \end{pmatrix},$$

$$V = \begin{pmatrix} 4\mathsf{e}^{3\lambda} - 2\mathsf{e}^{\lambda}v^2 & -4\mathsf{i}\mathsf{e}^{2\lambda}v + 2\mathsf{i}\mathsf{e}^{\lambda}\partial_x v - \mathsf{i}\partial_x^2 v + 2\mathsf{i}v^3 \\ 4\mathsf{i}\mathsf{e}^{2\lambda}v + 2\mathsf{i}\mathsf{e}^{\lambda}\partial_x v + \mathsf{i}\partial_x^2 v - 2\mathsf{i}v^3 & -4\mathsf{e}^{3\lambda} + 2\mathsf{e}^{\lambda}v^2 \end{pmatrix}.$$

A suitable (r, s) pair for this Lax pair is:

$$r(\lambda) = \frac{1}{2}\begin{pmatrix} \frac{-4}{\mathrm{e}^{\lambda}-\mathrm{e}^{-\lambda}} & 0 & 0 & \frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1} \\ 0 & 0 & -\frac{\mathrm{e}^{\lambda}+1}{\mathrm{e}^{\lambda}-1} & 0 \\ 0 & -\frac{\mathrm{e}^{\lambda}+1}{\mathrm{e}^{\lambda}-1} & 0 & 0 \\ \frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1} & 0 & 0 & \frac{-4}{\mathrm{e}^{\lambda}-\mathrm{e}^{-\lambda}} \end{pmatrix},$$

$$s(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

In broad terms, we have:

$$U + (r, s) \to \{\cdot, \cdot\}_S.$$

A suitable (r, s) pair for this Lax pair is:

$$r(\lambda) = \frac{1}{2}\begin{pmatrix} \frac{-4}{\mathrm{e}^{\lambda}-\mathrm{e}^{-\lambda}} & 0 & 0 & \frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1} \\ 0 & 0 & -\frac{\mathrm{e}^{\lambda}+1}{\mathrm{e}^{\lambda}-1} & 0 \\ 0 & -\frac{\mathrm{e}^{\lambda}+1}{\mathrm{e}^{\lambda}-1} & 0 & 0 \\ \frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1} & 0 & 0 & \frac{-4}{\mathrm{e}^{\lambda}-\mathrm{e}^{-\lambda}} \end{pmatrix},$$

$$s(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

However, this also works:

$$V + (r, s) \to \{\cdot, \cdot\}_T.$$

By introducing the fields $u = \partial_x v$ and $w = \partial_x^2 v$, combining the (r, s) pair with the V-matrix:

$$V = \begin{pmatrix} 4\mathrm{e}^{3\lambda} - 2\mathrm{e}^{\lambda} v^2 & -4\mathrm{i}\mathrm{e}^{2\lambda} v + 2\mathrm{i}\mathrm{e}^{\lambda} u - \mathrm{i} w + 2\mathrm{i} v^3 \\ 4\mathrm{i}\mathrm{e}^{2\lambda} v + 2\mathrm{i}\mathrm{e}^{\lambda} u + \mathrm{i} w - 2\mathrm{i} v^3 & -4\mathrm{e}^{3\lambda} + 2\mathrm{e}^{\lambda} v^2 \end{pmatrix},$$

we can find a non-ultralocal time-like Poisson structure:

$$\begin{aligned} \{v(t_1), v(t_2)\}_T = \{u(t_1), u(t_2)\}_T &= \{v(t_1), w(t_2)\}_T = 0, \\ \{v(t_1), u(t_2)\}_T &= \delta(t_1 - t_2), \\ \{u(t_1), w(t_2)\}_T &= -6v^2 \delta(t_1 - t_2), \\ \{w(t_1), w(t_2)\}_T &= \delta'(t_1 - t_2). \end{aligned}$$

and building the transfer matrix gives an equal-space Hamiltonian:

$$H_T = \frac{-1}{2} \int_{-\tau}^{\tau} (3v^4 - 2vw + u^2) \mathrm{d}t.$$

Summary:

- No reason to give the time-coordinate special treatment in (1+1)D models.
- Can provide a Hamiltonian/Poisson structure describing the “space evolution” of a system.
- Time-like boundary conditions can be extracted in the same manner as the space-like conditions.
- The equal-space approach seems to work for more complicated models (e.g. anisotropic Landau-Lifshitz) and non-ultralocal models (e.g. real mKdV).

Potential directions:

- Does the “lattice” commute?
- Find solutions to the higher order systems. Do they have any interesting consequences for the underlying systems?
- Implications and applications of time-like boundary conditions. Can space-like and time-like boundary conditions be combined to study space-time corners?
- A proper study of the anisotropic Landau-Lifshitz model or of non-ultralocal models.

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Thank you for listening!