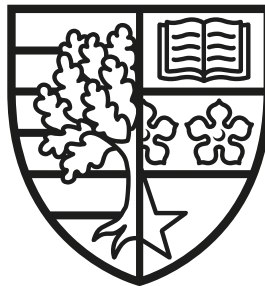# Integrable Hierarchies in the Lax/Zero-Curvature Formalism

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<small>SUBMITTED FOR THE DEGREE OF</small>
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Abstract

This thesis focusses on the development of (1+1)-dimensional integrable hierarchies in both the classical and quantum settings via the Lax/zero-curvature picture, where the underlying Poisson structure is found through the use of a classical or quantum R-matrix. After setting the scene by using the non-linear Schrödinger and isotropic Landau-Lifshitz models as examples of the standard approach to constructing hierarchies in this picture, the focus shifts to two more recent developments: equal-space Poisson structures (and the resulting spatially conserved quantities and Lax pairs); and quantum Lax pairs, where previously only the quantum Lax matrix (the spatial component) was considered. The non-linear Schrödinger and isotropic Landau-Lifshitz models (or analogous quantum spin chains) are then used as examples for these recent developments to compare against the familiar results.

For my family.

I may not visit, but I do still care.

Acknowledgements

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Citation details	A. Doikou, I. Findlay, "The quantum auxiliary linear problem & quantum Darboux-Bäcklund transformations", arXiv:1706.06052
Anastasia Doikou	Contributed an equal share of the ideas, writing, and proof-reading.
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Citation details	A. Doikou, I. Findlay, S. Sklaveniti, "Time-like boundary conditions in the NLS model", *Nucl. Phys.* **B941** (2019), 361-75, doi:10.1016/j.nuclphysb.2019.02.022
Anastasia Doikou	Contributed an equal share of the ideas, writing, and proof-reading.
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Contents

Chapter 1

Introduction

Integrable models play a fundamental role in many current topics, and some of the simplest physical examples of such are (1+1)-dimensional models, that is, those with one spatial coordinate x and one temporal coordinate t. Despite being labelled as "integrable", many of the examples of interest are very difficult or impractical to actually integrate, so it can be useful to develop connections between such models. Integrable hierarchies allow us to do just that, by describing a large (or even infinite in the case of systems with a continuous space dependence) number of integrable systems that are grouped and connected by some common object.

One method for constructing a hierarchy of integrable models is based on the existence of a "Lax matrix" (this can more generally be a Lax operator, but we focus in this thesis on the 2×2 matrix case), due originally to Lax [1] but later written in the zero-curvature approach we discuss here by others [2, 3], which allows the conserved quantities that provide the integrability, and the other integrable systems in the hierarchy, to be systematically constructed. This Lax matrix is used in conjunction with an R-matrix (for quantum systems, or its classical limit the r-matrix for classical systems), which is a solution of the Yang-Baxter equation [4, 5], (4.1.3) (or the classical Yang-Baxter equation, (2.1.6), for classical systems [6]). Together, the Lax matrix and R-matrix encode all of the relevant information of the integrable system.

When we talk about an integrable model, we typically mean a system of equations that govern the evolution of some fields. These fields depend on the spatial

coordinate (x for continuous systems and an index n for discrete systems) and the time coordinate t. For example, one of the two main models under study in this thesis is the non-linear Schrödinger (NLS) model[1] [2]:

$$\partial_t \psi = -\partial_x^2 \psi + 2\psi|\psi|^2, \qquad \partial_t \bar{\psi} = \partial_x^2 \bar{\psi} - 2\bar{\psi}|\psi|^2, \qquad (1.1.1)$$

which has two[2] fields, $\psi(x,t)$ and $\bar{\psi}(x,t)$. The derivative with respect to a variable is denoted by, for example, $\partial_x = \frac{\partial}{\partial x}$.

The other main model we concern ourselves with is the isotropic Landau-Lifshitz (HM) model[3] [7, 8], the equations of motion for which can be written in vector form:

$$\partial_t \vec{S} = \frac{\mathrm{i}}{c^2} \vec{S} \times (\partial_x^2 \vec{S}), \qquad (1.1.2)$$

where $\vec{S}(x,t) = \big(S_x(x,t), S_y(x,t), S_z(x,t)\big)^T$ is a vector containing the three fields $S_x(x,t)$, $S_y(x,t)$, and $S_z(x,t)$, and c is a non-zero constant, the meaning of which is given in (2.1.11). These can also be written in terms of useful combinations of these fields as:

$$\partial_t S_\pm = \pm\frac{1}{c^2}\big(S_\pm(\partial_x^2 S_z) - (\partial_x^2 S_\pm)S_z\big),$$

$$(1.1.3)$$

$$\partial_t S_z = \frac{1}{2c^2}\big((\partial_x^2 S_+)S_- - S_+(\partial_x^2 S_-)\big),$$

with $S_\pm(x,t) = S_x(x,t) \pm \mathrm{i}S_y(x,t)$. When referencing the three fields $S_\pm$ and S_z, we will use the subscript $\sigma \in \{+,-,z\}$ to collectively refer to them as S_σ.

Two other important ingredients in the extraction of these equations of motion are "Poisson brackets", $\{\cdot,\cdot\}$, and the "Hamiltonian", H. These are combined through Hamilton's equation to find the time-evolution of the fields in the model. For NLS, for example, the Poisson bracket and Hamiltonian would need to be chosen

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[1]This is usually presented with a factor of i on the left-hand side of each equation, so that the two are the complex conjugates of each other. For simplicity, we switch to imaginary time in this thesis so that the factors of i can be removed.

[2]While the ψ and $\bar{\psi}$ in these equations are not the complex conjugate of one another, as the over-bar would imply, we still retain the notation $|\psi|^2 = \psi\bar{\psi}$ to help keep the expressions compact.

[3]This model arises as the continuum limit of the classical analogue of the quantum XXX Heisenberg spin chain, which we shall discuss in Chapter 4. Due to the XXX spin chain's simple interpretation as a connected chain of magnets, it is also called the Heisenberg magnet, making the isotropic Landau-Lifshitz model the "continuous classical Heisenberg magnet", which is why we refer to it by HM.

such that the equations of motion, (1.1.1) could be written in the form of Hamilton's equation:

$$\partial_t \psi = \{H, \psi\}, \qquad \partial_t \bar{\psi} = \{H, \bar{\psi}\}. \qquad (1.1.4)$$

In the quantum case discussed in Chapter 4 the Poisson brackets will be replaced with commutation relations, and in the dual description of Chapter 3 the above equations would have ∂_x instead of ∂_t in the left-hand side, as we are interested in "space-evolution" there.

The Hamiltonian and Poisson bracket are of particular import, as a hallmark of integrability is the presence of a large number of distinct conserved quantities that Poisson commute with one another. Due to the anti-symmetry of the Poisson bracket, $\{a, b\} = -\{b, a\}$, the Hamiltonian is immediately one such quantity that is constant with respect to time. Then, part of the value of the Lax construction is that it provides a hierarchy of conserved quantities that Poisson commute with one another. Therefore, from the minimal input data of the Lax matrix and the r-matrix, we are able to extract all of the commuting quantities that we need for integrability.

A final important consideration when discussing solutions of models is that of boundary conditions. Two[4] types of boundary conditions will be discussed in this thesis: periodic and reflective boundary conditions.

As the simpler of the two cases, periodic boundary conditions will be imposed first in each setting, and the necessary tools for capturing what goes on at the boundary will be introduced after the periodic examples.

The rest of the thesis is laid out as follows: Chapter 2 walks through the process of how we classically use the tools of Lax matrices and r-matrices to determine the conserved quantities and temporal components of the Lax pairs for an integrable system, and how these provide us with the tower of related, yet distinct, integrable systems, each of which admits a description in this language. This will be done for

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[4]Despite this statement, all of the results for periodic boundary conditions will still hold in the case of Schwartz type boundary conditions (that is, those where the fields and all of their derivatives are assumed to vanish sufficiently quickly in the limit as $x \to \pm\infty$) by simply taking the limit as the length of the system goes to infinity.

both continuous (Sections 2.1-2.3) and discrete (Sections 2.4-2.6) models, both with periodic (Sections 2.2 and 2.5) and with reflective (Sections 2.3 and 2.6) boundary conditions. Examples of these results are provided in the continuous case for the NLS and HM models (which are fully introduced in this language in Section 2.1) and in the discrete case by the Ablowitz-Ladik model (which is a discretisation of the NLS model, and is introduced in Section 2.4). The results shown here are drawn from a selection of sources, with the primary reference being [9].

The next chapter, Chapter 3, is built from the papers [10] and [11] and builds the dual (equal-space[5]) construction of integrable models which was introduced in [12] and [13]. In this picture we describe models in terms of their space-evolution rather than the usual time-evolution, which leads to, for example, spatially conserved "Hamiltonians". The dual Poisson structure is introduced[6] in Section 3.1, and Section 3.2 generates the dual Hamiltonians and Lax pairs in the periodic case. Section 3.3 then investigates how we can use these to build a "lattice" in place of the usual hierarchy picture (although note, that the "lattice" does not necessarily commute). Finally, Section 3.4 introduces the idea of time-like boundary conditions in analogy to the space-like boundary conditions discussed in Chapter 2. The NLS and HM models are again used as examples throughout this chapter.

The penultimate chapter changes focus to instead study quantum spin chains, introducing the auxiliary linear problem to the quantum setting through the definition of the full quantum Lax pair. This chapter follows the work done in [14], but with additional examples using the Heisenberg XXX spin chain (as well as the quantum Ablowitz-Ladik model used in [14]). This brings the topic of quantum spin chains into closer comparison to classical discrete systems, as detailed in the latter half of Chapter 2.

Finally, we summarise the key points and make some closing remarks in the final chapter, Chapter 5.

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[5]This "equal-space" should be contrasted against the name "equal-time" for the standard picture described in Chapter 2. These names are in reference to the evolution variable, as they describe the coordinate that we are evolving along. I.e. stopping the evolution at some point would give us a space-time slice that has constant time (for equal-time/time-evolution) or constant space (for equal-space/space-evolution).

[6]Appendix A goes into some further properties of this dual Poisson structure and some of its consequences. Some more general results are also provided.

Chapter 2

Standard Hierarchies

To allow the later chapters to build on a common basis with consistent notation, this first chapter is devoted to introducing the models and tools that will be used throughout. It will also showcase how these are used to derive the quantities of interest so that the processes and results can be compared and contrasted with the appropriate analogues in the other chapters. We start by focussing on continuous integrable models, before turning to discrete models starting from Section 2.4.

For more details and references on this general construction, see [9] and [15].

2.1 Continuous Systems

2.1.1 The Equations of Interest

For both the NLS and HM models, we will use a dot to denote the derivative with respect to a time flow[1] t_k, e.g. $\dot{S}_\sigma = \partial_{t_k} S_\sigma$, and similarly use a prime to denote the derivative with respect to a space flow x_k, e.g. $S'_\sigma = \partial_{x_k} S_\sigma$. Where there is likely ambiguity however, we will explicitly use either ∂_{t_k} or ∂_{x_k}.

Both the NLS model [9, 16] and HM model [7, 8] can be written as the compatibility condition of the auxiliary linear problem:

$$\Psi' \equiv \partial_x \Psi = U\Psi, \qquad\qquad \dot{\Psi} \equiv \partial_t \Psi = V\Psi, \qquad (2.1.1)$$

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[1]These distinct time flows will arise from considering the tower of conserved quantities that define the system as integrable, and treating each of the quantities as the Hamiltonian for a distinct integrable system, describing the evolution of the fields along the associated time flow t_k. When we consider the dual picture in Chapter 3, we will likewise have a hierarchy of dual Hamiltonians that govern the space-evolution of the fields along a tower of space flows x_k.

where Ψ is an arbitrary vector field, and the 2×2 matrices U and V, depending on the fields in the system and some free complex parameter λ, comprise the Lax pair [1, 2, 3] for the system. For the NLS model, these are [15]:

$$U^{(\mathrm{NLS})} = \begin{pmatrix} \frac{\lambda}{2} & \bar{\psi} \\ \psi & -\frac{\lambda}{2} \end{pmatrix}, \qquad V^{(\mathrm{NLS})} = \begin{pmatrix} \frac{\lambda^2}{2} - |\psi|^2 & \lambda\bar{\psi} + \bar{\psi}' \\ \lambda\psi - \psi' & -\frac{\lambda^2}{2} + |\psi|^2 \end{pmatrix}, \qquad (2.1.2)$$

while for the HM model, these are [8]:

$$U^{(\mathrm{HM})} = \frac{1}{2\lambda}S, \qquad V^{(\mathrm{HM})} = \frac{1}{2\lambda^2}S - \frac{1}{2c^2\lambda}S'S, \qquad (2.1.3)$$

where:

$$S = \begin{pmatrix} S_z & S_- \\ S_+ & -S_z \end{pmatrix}. \qquad (2.1.4)$$

Cross-differentiating the auxiliary linear problem gives rise to the following compatibility condition (called the zero-curvature condition) between the matrices of the Lax pair:

$$0 = \dot{U} - V' + [U, V], \qquad (2.1.5)$$

such that when the matrices U and V are inserted into this, and the resulting equations are split about powers of λ, the appropriate equations of motion ((1.1.1) for (2.1.2) and (1.1.3) for (2.1.3)) are returned.

2.1.2 Poisson Brackets

The core objects in the Hamiltonian/r-matrix construction of integrable systems are the spatial component of the Lax pair, U, and an associated r-matrix that satisfies the classical Yang-Baxter equation [6]:

$$0 = [r_{ab}(\lambda - \mu), r_{ac}(\lambda)] + [r_{ab}(\lambda - \mu), r_{bc}(\mu)] + [r_{ac}(\lambda), r_{bc}(\mu)], \qquad (2.1.6)$$

where $\lambda, \mu \in \mathbb{C}$ are some free parameters and the subscripts denote which vector spaces the matrices act on (e.g. $r_{ab} = r \otimes \mathbb{I}$ and $r_{bc} = \mathbb{I} \otimes r$, with $r : V \otimes V \to V \otimes V$, so that the whole equation acts on $V_a \otimes V_b \otimes V_c$, where the subscripts attached to

the vector spaces are merely used to denote which index corresponds to them, e.g. r_{ab} would act only on the first two). Both the NLS and HM models share the same core r-matrix, but differ[2] by a factor of 2:

$$r^{(\mathrm{NLS})}(\lambda) = 2r^{(\mathrm{HM})}(\lambda) = \frac{1}{\lambda} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1.7}$$

These r-matrices are connected to the U-matrices and the defining equations for the system through the linear algebraic relation[3] [17]:

$$\{U_a(x, \lambda), U_b(y, \mu)\}_S = [r_{ab}(\lambda - \mu), U_a(x, \lambda) + U_b(y, \mu)]\, \delta(x - y), \tag{2.1.8}$$

which provides an ultra-local Poisson bracket between the fields. Inserting the U-matrices from (2.1.2) and (2.1.3) along with their respective r-matrices into this relation, we arrive at the Poisson brackets between the fields in each of the systems. For the NLS model, these are:

$$\{\psi(x), \bar{\psi}(y)\}_S = \delta(x - y), \tag{2.1.9}$$

while for the HM model, we find the $\mathfrak{sl}_2$ exchange relations:

$$\{S_\pm(x), S_z(y)\}_S = \pm S_\pm\, \delta(x - y), \qquad \{S_+(x), S_-(y)\}_S = -2S_z\, \delta(x - y). \tag{2.1.10}$$

The HM Poisson brackets accept a Casimir element, which manifests as a restriction of the vector $\vec{S}$ to the surface of the sphere of radius c, where we have labelled the Casimir c^2:

$$c^2 = S_z^2 + S_+ S_- = S_x^2 + S_y^2 + S_z^2. \tag{2.1.11}$$

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[2]Of course, any constant scalar factors can be removed by either scaling the Lax matrix or Hamiltonian, which leaves the equations of motion invariant. We choose to keep this extra factor, however, to keep the Lax pair and Hamiltonian consistent with the standard results.

[3]The subscript $_S$ is used here and in what follows to denote that we are building this system out of the <u>S</u>patial component of the Lax pair (U). This will be important later when we construct the dual model out of the <u>T</u>emporal component of the Lax pair (V), where we will use a $_T$ subscript.

2.2 Continuous Periodic Boundary Conditions

2.2.1 Conserved Quantities

In order to find conserved quantities that commute with respect to this Poisson bracket, we start by considering the (spatial) transport matrix, which is a path-ordered exponential solution to the spatial component of the auxiliary linear problem (2.1.1) in place of Ψ:

$$T_S(x, y; \lambda) = \mathrm{P}\exp\int_y^x U(\xi)\mathrm{d}\xi. \tag{2.2.1}$$

For a periodic system on the interval $[-L, L]$, i.e. where $\psi(L) = \psi(-L)$ and $S_\sigma(L) = S_\sigma(-L)$, the full monodromy matrix is $T_S(\lambda) = T_S(L, -L; \lambda)$. Due to the U-matrices satisfying the linear algebraic relation, (2.1.8), the monodromy matrix can be seen to satisfy a quadratic algebraic relation [18, 19]:

$$\{T_{S,a}(\lambda), T_{S,b}(\mu)\}_S = [r_{ab}(\lambda - \mu), T_{S,a}(\lambda)T_{S,b}(\mu)]. \tag{2.2.2}$$

Consequently, if we define a new object, called the transfer matrix $\mathfrak{t}_S(\lambda)$, as the trace of the monodromy matrix:

$$\mathfrak{t}_S(\lambda) = \mathrm{tr}\left\{T_S(\lambda)\right\}, \tag{2.2.3}$$

then this can be shown to Poisson commute with itself for different values of the spectral parameter (see, for example, [9]). Because of this, if we expand $\mathfrak{t}_S$ as a formal power series in λ, $\mathfrak{t}_S = \sum_k \lambda^k \mathfrak{t}_S^{(k)}$, then these coefficients also Poisson commute:

$$\{\mathfrak{t}_S^{(k)}, \mathfrak{t}_S^{(j)}\}_S = 0. \tag{2.2.4}$$

As such, the terms in this expansion $\mathfrak{t}_S^{(k)}$ can be seen as Hamiltonians[4] governing the evolution of the system along distinct time flows t_k. Further to this, the evolution along each time flow t_k will be integrable *à la* Liouville, as the $\mathfrak{t}_S^{(j)}$ with $j \neq k$ will provide the infinite tower of conserved quantities.

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[4]Although note that, as will be discussed below, these are non-local quantities, where we would rather have local Hamiltonians. This will be accounted for shortly.

The task is therefore to find the expansion of $\mathfrak{t}_S(\lambda)$ in some limit of λ. For the NLS Lax pair, (2.1.2), the appropriate limit is $\lambda \to \infty$, while for the HM Lax pair, (2.1.3), it is instead $\lambda \to 0^+$. In order to avoid evaluating the path-ordered exponential, we consider a diagonalisation of the transport matrix [9]:

$$T_S(x, y; \lambda) = \big(\mathbb{I} + W_S(x; \lambda)\big)\mathrm{e}^{Z_S(x,y;\lambda)}\big(\mathbb{I} + W_S(y; \lambda)\big)^{-1}, \qquad (2.2.5)$$

where W_S and Z_S are wholly anti-diagonal and diagonal matrices, respectively. If we insert this diagonalisation into the spatial half of the auxiliary linear problem, the diagonal and anti-diagonal components can be separated into two relations:

$$Z_S' = U_D + U_A W_S,$$

$$0 = W_S' + [W_S, U_D] + W_S U_A W_S - U_A, \qquad (2.2.6)$$

where U_D and U_A are the diagonal and anti-diagonal components of the U-matrix, respectively. We now expand W_S and Z_S in powers of λ, with coefficients $W_S^{(k)}$ and $Z_S^{(k)}$ [9]. For the NLS model, these expansions will be:

$$W_S(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k} W_S^{(k)}, \qquad\qquad Z_S(\lambda) = \sum_{k=-1}^{\infty} \lambda^{-k} Z_S^{(k)}, \qquad (2.2.7)$$

while for the HM model these expansions will swap $\lambda^{-1} \to \lambda$. Using these, we can split (2.2.6) into a series of recurrence relations which we can recursively solve to find ever higher coefficients in the series expansions of W_S and Z_S. For the NLS model, the first few terms in the Z_S-series are relatively simple:

$$Z_S^{(\mathrm{NLS},-1)} = L\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Z_S^{(\mathrm{NLS},0)} = 0, \qquad (2.2.8)$$

$$Z_S^{(\mathrm{NLS},1)} = \int_{-L}^{L} |\psi|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{d}x,$$

although the terms that follow them grow increasingly complicated:

$$Z_S^{(\text{NLS},2)} = \int_{-L}^{L} \begin{pmatrix} -\psi'\bar{\psi} & 0 \\ 0 & \psi\bar{\psi}' \end{pmatrix} \mathrm{d}x,$$

$$Z_S^{(\text{NLS},3)} = \int_{-L}^{L} \begin{pmatrix} \psi''\bar{\psi} - |\psi|^4 & 0 \\ 0 & |\psi|^4 - \psi\bar{\psi}'' \end{pmatrix} \mathrm{d}x, \tag{2.2.9}$$

$$Z_S^{(\text{NLS},4)} = \int_{-L}^{L} \begin{pmatrix} \psi\bar{\psi}' + 4\psi'\bar{\psi} - \psi'''\bar{\psi} & 0 \\ 0 & \psi'\bar{\psi} + 4\psi\bar{\psi}' - \psi\bar{\psi}''' \end{pmatrix} \mathrm{d}x,$$

while for the HM model, they are:

$$Z_S^{(\text{HM},-1)} = cL \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Z_S^{(\text{HM},0)} = \frac{1}{4c} \int_{-L}^{L} \frac{S_+ S_-' - S_+' S_-}{c + S_z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{d}x, \tag{2.2.10}$$

$$Z_S^{(\text{HM},1)} = \frac{-1}{4c^3} \int_{-L}^{L} \left(S_+' S_-' + (S_z')^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{d}x.$$

The reason for doing this is that if we insert the decomposition of the monodromy matrix, (2.2.5), into the definition of the transfer matrix, (2.2.3), we can use the cyclic property of the trace and the periodicity of the system we are considering to cancel the explicit W dependence, leaving:

$$\mathsf{t}_S(\lambda) = \operatorname{tr}\left\{ e^{Z_S(\lambda)} \right\}$$

$$= e^{Z_{11,S}(\lambda)} + e^{Z_{22,S}(\lambda)}. \tag{2.2.11}$$

As any combination of conserved commuting quantities will also be a conserved commuting quantity, it is useful to instead consider the expansion of the logarithm of the transfer matrix, $\mathcal{G}_S(\lambda) = \ln\left(\mathsf{t}_S(\lambda)\right)$, as this acts to remove the non-locality of

the conserved quantities. The expansion of this logarithmic generator is then:

$$\mathcal{G}_S^{(\text{NLS})}(\lambda) = \ln\left(e^{\lambda Z_{11,S}^{(\text{NLS},-1)}+Z_{11,S}^{(\text{NLS},0)}+\cdots} + e^{\lambda Z_{22,S}^{(\text{NLS},-1)}+Z_{22,S}^{(\text{NLS},0)}+\cdots}\right),$$

$$\mathcal{G}_S^{(\text{HM})}(\lambda) = \ln\left(e^{\lambda^{-1} Z_{11,S}^{(\text{HM},-1)}+Z_{11,S}^{(\text{HM},0)}+\cdots} + e^{\lambda^{-1} Z_{22,S}^{(\text{HM},-1)}+Z_{22,S}^{(\text{HM},0)}+\cdots}\right). \tag{2.2.12}$$

The leading order contribution $Z_S^{(-1)}$ from both the NLS Z-matrices, (2.2.8), and the HM Z-matrices, (2.2.10), are such that when the appropriate λ limit is taken, the (1, 1)-component will approach $+\infty$ while the (2, 2)-component will approach $-\infty$. Consequently, the exponential of the (1, 1)-components will dominate in these limits, so that the expansion of $\mathcal{G}_S(\lambda)$ is:

$$\mathcal{G}_S^{(\text{NLS})}(\lambda) = \lambda Z_{11,S}^{(\text{NLS},-1)} + Z_{11,S}^{(\text{NLS},0)} + \lambda^{-1} Z_{11,S}^{(\text{NLS},1)} + ...,$$

$$\mathcal{G}_S^{(\text{HM})}(\lambda) = \lambda^{-1} Z_{11,S}^{(\text{HM},-1)} + Z_{11,S}^{(\text{HM},0)} + \lambda Z_{11,S}^{(\text{HM},1)} + ... \tag{2.2.13}$$

For the NLS model, we can combine this with the Z-series, (2.2.8) and (2.2.9), to find the first six conserved quantities:

$$\mathcal{G}_S^{(\text{NLS},-1)} = L,$$

$$\mathcal{G}_S^{(\text{NLS},0)} = 0,$$

$$\mathcal{G}_S^{(\text{NLS},1)} = \int_{-L}^{L} |\psi|^2 \mathrm{d}x,$$

$$\mathcal{G}_S^{(\text{NLS},2)} = -\int_{-L}^{L} \psi'\bar{\psi}\mathrm{d}x,$$

$$\mathcal{G}_S^{(\text{NLS},3)} = \int_{-L}^{L} \left(\psi''\bar{\psi} - |\psi|^4\right)\mathrm{d}x,$$

$$\mathcal{G}_S^{(\text{NLS},4)} = \int_{-L}^{L} \left(\psi\bar{\psi}' + 4\psi'\bar{\psi} - \psi'''\bar{\psi}\right)\mathrm{d}x, \tag{2.2.14}$$

a few of which are known to have direct physical interpretations: $\mathcal{G}_S^{(\text{NLS},1)}$ is the total density of the system, $\mathcal{G}_S^{(\text{NLS},2)}$ is the total momentum of the system, $\mathcal{G}_S^{(\text{NLS},3)}$ is the Hamiltonian for the NLS model, and $\mathcal{G}_S^{(\text{NLS},4)}$ is the Hamiltonian for the complex

modified Korteweg-de Vries (cmKdV) model:

$$N_S^{(\text{NLS})} = \mathcal{G}_S^{(\text{NLS},1)}, \qquad\qquad P_S^{(\text{NLS})} = \mathcal{G}_S^{(\text{NLS},2)},$$

$$\tag{2.2.15}$$

$$H_S^{(\text{NLS})} = \mathcal{G}_S^{(\text{NLS},3)}, \qquad\qquad H_S^{(\text{cmKdV})} = \mathcal{G}_S^{(\text{NLS},4)}.$$

For the HM model, the $\mathcal{G}_S$ expansion causes (2.2.10) to supply us with the first three conserved quantities:

$$\mathcal{G}_S^{(\text{HM},-1)} = cL,$$

$$\mathcal{G}_S^{(\text{HM},0)} = \frac{1}{4c} \int_{-L}^{L} \frac{S_+ S_-' - S_+' S_-}{c + S_z} \, \mathrm{d}x, \tag{2.2.16}$$

$$\mathcal{G}_S^{(\text{HM},1)} = \frac{-1}{4c^3} \int_{-L}^{L} \left(S_+' S_-' + (S_z')^2 \right) \mathrm{d}x,$$

the second and third of which can be recognised as the total momentum and Hamiltonian for the HM model, respectively (up to a factor of $-2c$) [9]:

$$P_S^{(\text{HM})} = -2c\mathcal{G}_S^{(\text{HM},0)}, \qquad\qquad H_S^{(\text{HM})} = -2c\mathcal{G}_S^{(\text{HM},1)}. \tag{2.2.17}$$

2.2.2 V-Matrices

Each of the conserved quantities $\mathcal{G}_S^{(k)}$ generated through the expansion of $\mathcal{G}_S$ can be seen to describe the evolution of the system along a distinct time flow t_k, so that the equations of motion for each of these systems would be given by:

$$\partial_{t_k} \mathfrak{f} = \{\mathcal{G}_S^{(k)}, \mathfrak{f}\}_S, \tag{2.2.18}$$

where $\mathfrak{f}$ is a field of the model under study (e.g. $\mathfrak{f} = \psi$ or $\mathfrak{f} = \bar{\psi}$ for NLS). Consequently, each of these systems should have some associated Lax pair. As we use the U-matrix to generate the conserved quantities we will be looking for a generator $\mathbb{V}$ that produces the V-matrices $V^{(k)}$ associated to each time flow t_k. We do so by first

equating Hamilton's equation (as applied to U) and the zero-curvature condition:

$$\mathbb{V}_b'(\lambda, \mu) - [U_b(\lambda), \mathbb{V}_b(\lambda, \mu)] = \partial_{\bar{t}} U_b(\lambda) = \left\{ \ln \, \text{tr}_a\{T_{S,a}(\mu)\}, U_b(\lambda) \right\}_S$$

$$= \mathfrak{t}_S^{-1}(\mu) \text{tr}_a\{T_{S,a}(\mu), U_b(\lambda)\}_S, \tag{2.2.19}$$

where the $\bar{t}$ is used to denote some master time flow and the vector space subscripts are introduced to distinguish the space being traced over (the a vector space). Using the algebraic relations (2.1.8) and (2.2.2), we can extract from this the generator of the V-matrices associated to each time flow t_k, [6]:

$$\mathbb{V}_b(x; \lambda, \mu) = \mathfrak{t}_S^{-1}(\mu) \, \text{tr}_a \left\{ T_{S,a}(L, x; \mu) r_{ab}(\mu - \lambda) T_{S,a}(x, -L; \mu) \right\}, \tag{2.2.20}$$

such that the V-matrix associated to the t_k time flow appears as the coefficient of μ^{-k} (for the NLS model) or μ^k (for the HM model) in the series expansion of this about μ. Using the diagonalisation of the monodromy matrix, the suitable limit of μ in $Z(\mu)$, and the cyclic properties of the trace, this can be simplified to:

$$\mathbb{V}_b(x; \lambda, \mu) = \text{tr}_a \left\{ r_{ab}(\mu - \lambda) \left(\mathbb{I} + W_{S,a}(x; \mu) \right) e_{11,a} \left(\mathbb{I} + W_{S,a}(x; \mu) \right)^{-1} \right\}, \tag{2.2.21}$$

where e_{ij} is the 2×2 matrix that obeys $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Finally, as the chosen r-matrix satisfies the property $r_{ab}M_a = M_b r_{ab}$ for any 2×2 matrix M, this can be simplified further to lie solely in the b vector space (so that we may drop the subscripts):

$$\mathbb{V}^{(\text{NLS})}(x; \lambda, \mu) = 2\mathbb{V}^{(\text{HM})}(x; \lambda, \mu) = \frac{1}{\mu - \lambda} \left(\mathbb{I} + W_S(x; \mu) \right) e_{11} \left(\mathbb{I} + W_S(x; \mu) \right)^{-1}. \tag{2.2.22}$$

Using the W-matrices found by inserting the NLS U-matrix into (2.2.6), the limit $\lambda \to \infty$ supplies the V-matrices corresponding to the conserved quantities stated in (2.2.14). As a trivial Hamiltonian will provide trivial equations of motion, the V-matrices corresponding to any trivial conserved quantities will themselves be trivial. Consequently, the series of V-matrices for the NLS model starts at order 1:

$$\mathbb{V}^{(\text{NLS},1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.2.23}$$

and continues to higher orders:

$$\mathbb{V}^{(\text{NLS},2)} = \begin{pmatrix} \lambda & \bar{\psi} \\ \psi & 0 \end{pmatrix},$$

$$\mathbb{V}^{(\text{NLS},3)} = \begin{pmatrix} \lambda^2 - |\psi|^2 & \lambda\bar{\psi} + \bar{\psi}' \\ \lambda\psi - \psi' & |\psi|^2 \end{pmatrix}, \tag{2.2.24}$$

$$\mathbb{V}^{(\text{NLS},4)} = \begin{pmatrix} \lambda^3 - \lambda|\psi|^2 + (\psi'\bar{\psi} - \psi\bar{\psi}') & \lambda^2\bar{\psi} + \lambda\bar{\psi}' + (\bar{\psi}'' - 2\bar{\psi}|\psi|^2) \\ \lambda^2\psi - \lambda\psi' + (\psi'' - 2\psi|\psi|^2) & \lambda|\psi|^2 - (\psi'\bar{\psi} - \psi\bar{\psi}') \end{pmatrix}.$$

The V-matrix presented in the original NLS Lax pair, (2.1.2), is then recognised as the third-order term, up to a shifting by some constant factor:

$$V^{(\text{NLS})} = \mathbb{V}^{(\text{NLS},3)} - \frac{\lambda^2}{2}\mathbb{I}. \tag{2.2.25}$$

If we instead use the HM U-matrix in (2.2.6), and expand (2.2.22) in the limit $\mu \to 0^+$, the first three terms are:

$$\mathbb{V}^{(\text{HM},0)} = \frac{-1}{4\lambda}\mathbb{I} - \frac{1}{4c\lambda}S,$$

$$\mathbb{V}^{(\text{HM},1)} = \frac{-1}{4\lambda^2}\mathbb{I} - \frac{1}{4c\lambda^2}S + \frac{1}{4c^3\lambda}S'S, \tag{2.2.26}$$

$$\mathbb{V}^{(\text{HM},2)} = \frac{-1}{4\lambda^3}\mathbb{I} - \frac{1}{4c\lambda^3}S + \frac{1}{4c^3\lambda^2}S'S - \frac{1}{4c^3\lambda}S'' - \frac{3}{8c^5\lambda}(S')^2 S.$$

After removing the overall commuting constant factors and scaling by $-2c$, the second of these can be identified as the V-matrix in the original HM Lax pair (2.1.3):

$$V^{(\text{HM})} = -2c(\mathbb{V}^{(\text{HM},1)} + \frac{1}{4\lambda^2}\mathbb{I}). \tag{2.2.27}$$

It is the identifications of $U^{(\text{NLS})}$ with $\mathbb{V}^{(\text{NLS},2)}$ and $U^{(\text{HM})}$ with $\mathbb{V}^{(\text{HM},0)}$ (up to some constant factors), that suggest the introduction of dual pictures for these models, with the roles of time and space switched. This is addressed in Chapter 3. Before we do that though, we discuss the changes that need to be made to this construction

to account for having non-periodic boundary conditions, as well as how this picture fits to models with discrete space dependence.

2.3 Continuous Open Boundary Conditions

2.3.1 Conserved Quantities

In order to study systems with open boundary conditions, we need to introduce some $K_\pm$-matrices that are associated to the $\pm L$ boundaries, and have a dependence on some boundary fields and an additional spectral parameter. In order for them to be used in generating conserved quantities, we require that they satisfy the classical analogue of the (static) quantum reflection equation [20, 21]:

$$
\begin{aligned}
0 = {} & K_{\pm,a}(\lambda) r_{ab}(\lambda + \mu) K_{\pm,b}(\mu) - K_{\pm,b}(\mu) r_{ab}(\lambda + \mu) K_{\pm,a}(\lambda) \\
& + [r_{ab}(\lambda - \mu), K_{\pm,a}(\lambda) K_{\pm,b}(\mu)].
\end{aligned}
\tag{2.3.1}
$$

For our choices of r-matrix, (2.1.7), the most general choice of $K_\pm$-matrix (up to some rescaling and gauge transformations) is [22]:

$$
K_\pm^{(\mathrm{NLS})}(\lambda) = K_\pm^{(\mathrm{HM})}(\lambda) = \alpha_\pm \mathbb{I} + \lambda \begin{pmatrix} \delta_\pm & \beta_\pm \\ \gamma_\pm & -\delta_\pm \end{pmatrix},
\tag{2.3.2}
$$

where $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, and $\delta_\pm$ are some constants that describe the boundary conditions being considered[5]. In this thesis we only consider the non-dynamical case where the constants have no time dependence[6] (and when we move on to discuss time-like boundary conditions, we shall assume that the equivalent constants have no space dependence).

These $K_\pm$-matrices are introduced into the transfer matrix $\mathfrak{t}_S$ as [20]:

$$
\bar{\mathfrak{t}}_S(\lambda) = \mathrm{tr}\left\{ K_+(\lambda) T_S(L, -L; \lambda) K_-(\lambda) T_S^{-1}(L, -L; -\lambda) \right\},
\tag{2.3.3}
$$

---

[5]The reflection equations satisfied by K_+ and K_- actually differ by a minus sign in the spectral parameter, but this factor can be absorbed into the β_+, γ_+, and δ_+ so that the boundary matrices both take the same form.

[6]Introducing a time-dependence into the K-matrix would result in a Poisson bracket between the K-matrices on the left-hand side of (2.3.1), in the manner of (2.1.8) or (2.2.2).

and from this definition it follows that:

$$\{\bar{\mathfrak{t}}_S(\lambda), \bar{\mathfrak{t}}_S(\mu)\}_S = 0. \tag{2.3.4}$$

Much as in the periodic case, we will consider the generator $\bar{\mathcal{G}}_S(\lambda) = \ln\left(\bar{\mathfrak{t}}_S(\lambda)\right)$, as this will supply us with the known Hamiltonian. When we diagonalise the T_S^{-1}, we use:

$$T_S^{-1}(x, y; -\lambda) = \left(\mathbb{I} + W_S(y; -\lambda)\right)\mathrm{e}^{-Z_S(x,y;-\lambda)}\left(\mathbb{I} + W_S(x; -\lambda)\right)^{-1}, \tag{2.3.5}$$

instead of (2.2.5). Consequently, as the highest order term in Z_S is λ for the NLS model and λ^{-1} for the HM model, the effect of the $-$ sign outside of the Z_S and the change in sign of the λ will cancel out, so that the appropriate limit of the exponential term will be:

$$\mathrm{e}^{-Z_S^{(\mathrm{NLS})}(x,y;-\lambda)} \rightarrow \mathrm{e}^{-Z_{11,S}^{(\mathrm{NLS})}(x,y;-\lambda)}e_{11} + \mathcal{O}(\mathrm{e}^{-\lambda^{-1}}),$$
$$\mathrm{e}^{-Z_S^{(\mathrm{HM})}(x,y;-\lambda)} \rightarrow \mathrm{e}^{-Z_{11,S}^{(\mathrm{HM})}(x,y;-\lambda)}e_{11} + \mathcal{O}(\mathrm{e}^{-\lambda}). \tag{2.3.6}$$

Consequently, the expansion of the generator $\bar{\mathcal{G}}_S$ will be:

$$\bar{\mathcal{G}}_S(\lambda) = Z_{11,S}(\lambda) - Z_{11,S}(-\lambda) + \ln\left(\left[\left(\mathbb{I} + W_S(L; -\lambda)\right)^{-1}K_+(\lambda)\left(\mathbb{I} + W_S(L; \lambda)\right)\right]_{11}\right)$$
$$+ \ln\left(\left[\left(\mathbb{I} + W_S(-L; \lambda)\right)^{-1}K_-(\lambda)\left(\mathbb{I} + W_S(-L; -\lambda)\right)\right]_{11}\right), \tag{2.3.7}$$

where the $[...]_{ij}$ indicates that we are only considering the ijth component of the matrix inside the brackets. Due to the $Z_{11,S}(\lambda) - Z_{11,S}(-\lambda)$ comprising the bulk Hamiltonian, it immediately follows that all of the conserved quantities that appeared at even order in (2.2.14) (namely, $\mathcal{G}_S^{(\mathrm{NLS},2)}$ and $\mathcal{G}_S^{(\mathrm{NLS},4)}$) and (2.2.16) (namely, $\mathcal{G}_S^{(\mathrm{HM},0)}$) will be 0. The remaining non-trivial NLS conserved quantities from (2.2.14)

are given by [23]:

$$\bar{\mathcal{G}}_S^{(\text{NLS},1)} = 2\int_{-L}^{L} |\psi|^2 \mathrm{d}x + \frac{1}{\delta_+}\left(\alpha_+ + \beta_+\psi - \gamma_+\bar{\psi}\right)\big|_{x=L}$$

$$+ \frac{1}{\delta_-}\left(\alpha_- - \beta_-\psi + \gamma_-\bar{\psi}\right)\big|_{x=-L}, \quad (2.3.8)$$

$$\bar{\mathcal{G}}_S^{(\text{NLS},3)} = 2\int_{-L}^{L} \left(\psi''\bar{\psi} - |\psi|^4\right)\mathrm{d}x + \mathcal{B}_+(L) - \mathcal{B}_-(-L),$$

where:

$$\mathcal{B}_\pm = -2\bar{\psi}\psi' + \frac{1}{\delta_\pm}\left(\beta_\pm(\psi'' - 2\psi|\psi|^2) - \gamma_\pm(\bar{\psi}'' - 2\bar{\psi}|\psi|^2) \mp 2\alpha_\pm|\psi|^2\right)$$

$$(2.3.9)$$

$$+ \frac{1}{3\delta_\pm^2}\left(\beta_\pm\psi - \gamma_\pm\bar{\psi} \pm \alpha_\pm\right)\left(\beta_\pm\psi - \gamma_\pm\bar{\psi}\right)',$$

while the remaining non-trivial HM conserved quantity from (2.2.16) is [24]:

$$\bar{\mathcal{G}}_S^{(\text{HM},1)} = \frac{-1}{2c^3}\int_{-L}^{L}\left(S_+'S_-' + (S_z')^2\right)\mathrm{d}x + \frac{1}{2\alpha_+ c}\left[2\delta_+ S_z + \beta_+ S_+ + \gamma_+ S_-\right]_{x=+L}$$

$$+ \frac{1}{2\alpha_- c}\left[2\delta_- S_z + \beta_- S_+ + \gamma_- S_-\right]_{x=-L}.$$

$$(2.3.10)$$

These remaining conserved quantities can be recognised as $\mathcal{G}_S^{(\text{NLS},1)}$ and $\mathcal{G}_S^{(\text{NLS},3)}$ from (2.2.14) and $\mathcal{G}_S^{(\text{HM},1)}$ from (2.2.16), up to boundary contributions and an overall factor of 2. As $\mathcal{G}_S^{(\text{NLS},2)}$ and $\mathcal{G}_S^{(\text{HM},0)}$ were associated to the total momentums of their respective systems, and are trivial in this setting, we can infer that the momentum is no longer conserved in either model when boundary conditions are introduced.

2.3.2 V-Matrices

By following an analogous derivation to that of (2.2.20), we can derive the generator of the V-matrices corresponding to the conserved quantities generated by $\bar{\mathcal{G}}_S$. There are three cases to consider in this setting [25], corresponding to the V-matrices in the bulk (labelled $\bar{\mathbb{V}}_\text{B}$), and the V-matrices lying at each of the two boundaries (labelled $\bar{\mathbb{V}}_\pm$ for the $x = \pm L$ boundaries). The generator of the bulk V-matrices is

(the parameter dependence on the left-hand side is suppressed for compactness):

$$\bar{\mathbb{V}}_{\mathrm{B},b} = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_a \left\{ K_{+,a}(\mu)T_{S,a}(L,x;\mu)r_{ab}(\mu-\lambda)T_{S,a}(x,-L;\mu)K_{-,a}(\mu)T_{S,a}^{-1}(-\mu) \right.$$

$$\left. + K_{+,a}(\mu)T_{S,a}(\mu)K_{-,a}(\mu)T_{S,a}^{-1}(x,-L;-\mu)r_{ab}(\mu+\lambda)T_{S,a}^{-1}(L,x;-\mu) \right\},$$

$$(2.3.11)$$

while the generator of the V-matrices at the positive boundary is:

$$\bar{\mathbb{V}}_{+,b}(\lambda,\mu) = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_a \left\{ K_{-,a}(\mu)T_{S,a}^{-1}(-\mu)K_{+,a}(\mu)r_{ab}(\mu-\lambda)T_{S,a}(\mu) \right.$$

$$(2.3.12)$$

$$\left. + K_{-,a}(\mu)T_{S,a}^{-1}(-\mu)r_{ab}(\mu+\lambda)K_{+,a}(\mu)T_{S,a}(\mu) \right\},$$

and the generator of the V-matrices at the negative boundary is:

$$\bar{\mathbb{V}}_{-,b}(\lambda,\mu) = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_a \left\{ K_{+,a}(\mu)T_{S,a}(\mu)r_{ab}(\mu-\lambda)K_{-,a}(\mu)T_{S,a}^{-1}(-\mu) \right.$$

$$(2.3.13)$$

$$\left. + K_{+,a}(\mu)T_{S,a}(\mu)K_{-,a}(\mu)r_{ab}(\mu+\lambda)T_{S,a}^{-1}(-\mu) \right\}.$$

If we expand these three generators in the appropriate limit of μ, the even ordered terms are all trivial ($\mathbb{V}^{(\mathrm{NLS},2)}$ and $\mathbb{V}^{(\mathrm{NLS},4)}$ in (2.2.24) and $\mathbb{V}^{(\mathrm{HM},0)}$ and $\mathbb{V}^{(\mathrm{HM},2)}$ in (2.2.26)), corresponding to how the $\bar{\mathcal{G}}_S^{(\mathrm{even})}$ were constant. In the limit as $\mu \to \infty$ of the generator of the NLS V-matrices, the remaining non-trivial terms from (2.2.24) are the order μ^3 matrix (which is the one we are interested in), where we define $\xi_\pm = \beta_\pm \psi - \gamma_\pm \bar{\psi}$, split here into its bulk and boundary contributions:

$$\bar{\mathbb{V}}_{\mathrm{B}}^{(\mathrm{NLS},3)}(x;\lambda) = 2\begin{pmatrix} \lambda^2 - |\psi|^2 & \lambda\bar{\psi} + \bar{\psi}' \\ \lambda\psi - \psi' & |\psi|^2 \end{pmatrix},$$

$$\bar{\mathbb{V}}_{\pm}^{(\mathrm{NLS},3)}(\lambda) = 2\begin{pmatrix} \lambda^2 - |\psi|^2 & \lambda\bar{\psi} \\ \lambda\psi & |\psi|^2 \end{pmatrix} + \frac{1}{\delta_\pm^3}\left((\alpha_\pm \pm \xi_\pm)^2 \pm \xi_\pm'\right)\begin{pmatrix} 0 & \beta_\pm \\ \gamma_\pm & 0 \end{pmatrix}$$

$$(2.3.14)$$

$$\pm \frac{1}{\delta_\pm^2}(\alpha_\pm \pm \xi_\pm)\begin{pmatrix} \xi_\pm & -\lambda\beta_\pm \\ \lambda\gamma_\pm & -\xi_\pm \end{pmatrix} \pm \frac{1}{\delta_\pm^2}\xi_\pm'\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$- \frac{1}{\delta_\pm}\begin{pmatrix} \beta_\pm\psi + \gamma_\pm\bar{\psi} + \xi_\pm' & -\lambda^2\beta_\pm + 2\beta_\pm|\psi|^2 \pm 2\alpha_\pm\bar{\psi} \\ -\lambda^2\gamma_\pm + 2\gamma_\pm|\psi|^2 \mp 2\alpha_\pm\psi & -(\beta_\pm\psi + \gamma_\pm\bar{\psi} + \xi_\pm') \end{pmatrix},$$

and the order μ^1 matrix (also split into bulk and boundary terms):

$$\bar{\mathbb{V}}_{\text{B}}^{(\text{NLS},1)}(x;\lambda) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{\mathbb{V}}_{\pm}^{(\text{NLS},1)}(\lambda) = \frac{1}{\delta_\pm}\begin{pmatrix} 0 & \beta_\pm \\ \gamma_\pm & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3.15}$$

When the HM model is considered instead, the limit as $\mu \to 0^+$ of these generators gives the only non-trivial term up to order μ^2 (to match with (2.2.26)) as:

$$\bar{\mathbb{V}}_{\text{B}}^{(\text{HM},1)}(x;\lambda) = \frac{-1}{2\lambda^2}\mathbb{I} - \frac{1}{2c\lambda^2}S + \frac{1}{2c^3\lambda}S'S, \tag{2.3.16}$$

$$\bar{\mathbb{V}}_{\pm}^{(\text{HM},1)}(\lambda) = \frac{-1}{2\lambda^2}\mathbb{I} - \frac{1}{2c\lambda^2}S \pm \frac{1}{4\alpha_\pm c\lambda}\begin{pmatrix} \beta_\pm S_+ - \gamma_\pm S_- & 2(\delta_\pm S_- - \beta_\pm S_z) \\ 2(\gamma_\pm S_z - \delta_\pm S_+) & \gamma_\pm S_- - \beta_\pm S_+ \end{pmatrix}.$$

In order to extract the boundary conditions from the open Hamiltonian, we simply calculate the equations of motion as usual (through the Poisson brackets and Hamilton's equation), except gathering all of the boundary terms that arise (either from the integration of total derivatives in the bulk Hamiltonian, or from the Poisson bracket of the fields with the boundary Hamiltonians). We then impose the sewing conditions that the equations of motion away from the boundary smoothly transition to those at the boundary, i.e. that $\lim_{x\to\pm L}\dot{\psi}(x) = \dot{\psi}(\pm L)$ and $\lim_{x\to\pm L}\dot{\bar{\psi}}(x) = \dot{\bar{\psi}}(\pm L)$ for the NLS model and that $\lim_{x\to\pm L}\dot{S}_\sigma(x) = \dot{S}_\sigma(\pm L)$ for the HM model.

Similarly, in order to extract the boundary conditions from the V-matrices, the condition that the equations of motion agree at the boundary manifests as the condition that $\lim_{x\to\pm L}\bar{\mathbb{V}}_{\text{B},b} = \bar{\mathbb{V}}_{\pm,b}$. Performing either of these limits yields the same constraints on the fields, so they can be used as a consistency check for one another. The boundary conditions that arise (through either of these methods) for the NLS model are [23]:

$$\beta_\pm = \gamma_\pm = 0, \qquad \psi'(\pm L) = \mp\frac{\alpha_\pm}{\delta_\pm}\psi, \qquad \bar{\psi}'(\pm L) = \mp\frac{\alpha_\pm}{\delta_\pm}\bar{\psi}, \qquad (2.3.17)$$

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while the conditions on the S_σ in the HM model are [24]:

$$\left[S_+ S'_- - S'_+ S_-\right]_{x=\pm L} = \pm \frac{c^2}{\alpha_\pm}\left[\beta_\pm S_+ - \gamma_\pm S_-\right]_{x=\pm L},$$

$$\left[S_+ S'_z - S'_+ S_z\right]_{x=\pm L} = \pm \frac{c^2}{\alpha_\pm}\left[\delta_\pm S_+ - \gamma_\pm S_z\right]_{x=\pm L}, \qquad (2.3.18)$$

$$\left[S_- S'_z - S'_- S_z\right]_{x=\pm L} = \pm \frac{c^2}{\alpha_\pm}\left[\delta_\pm S_- - \beta_\pm S_z\right]_{x=\pm L}.$$

2.4 Discrete Systems

2.4.1 The Equations of Interest

One of the key results covered in this thesis is the construction of the auxiliary linear problem in the quantum spin chain setting in Chapter 4, with a suitable generator of the temporal components of the Lax pair. Consequently, we briefly cover the known results for classical semi-discrete systems. In Chapter 4 we discuss the Heisenberg XXX spin chain and the quantum Ablowitz-Ladik model [26], which are discrete quantum versions of the non-linear Schrödinger and isotropic Landau-Lifshitz equations covered in Section 2.1. To connect these two pictures, we use the classical Ablowitz-Ladik (AL) model [27] to demonstrate the key points of this section[7].

The AL model is governed by the evolution equations [27]:

$$\partial_t \psi_n = 2\psi_n - \psi_{n+1} - \psi_{n-1} + |\psi_n|^2(\psi_{n+1} + \psi_{n-1}),$$

$$\qquad (2.4.1)$$

$$\partial_t \bar{\psi}_n = -2\bar{\psi}_n + \bar{\psi}_{n+1} + \bar{\psi}_{n-1} - |\psi_n|^2(\bar{\psi}_{n+1} + \bar{\psi}_{n-1}),$$

which, in the limit $\psi_n \to \Delta\psi(\Delta n)$ and $t \to \Delta^2 t$, yield the continuous non-linear Schrödinger equations (1.1.1).

---

[7]The discrete isotropic Landau-Lifshitz equation is not discussed here due to requiring a selection of tricks that are needed for none of the continuous (Section 2.1), dual (Chapter 3), or quantum (Chapter 4) constructions, so doesn't serve as a useful connection.

Just as in the continuous case, these equations arise as the compatibility condition of a pair of equations, a discrete analogue of the auxiliary linear problem, (2.1.1), which is also called the auxiliary linear problem (the context in which they are referenced will be sufficient to determine whether the continuous or discrete version is being considered):

$$\Psi_{n+1} = L_n \Psi_n, \qquad\qquad \dot{\Psi}_n \equiv \partial_t \Psi_n = A_n \Psi_n. \qquad (2.4.2)$$

The matrices (L_n, A_n) are called the discrete Lax pair[8] for an integrable system if the equations of motion for that system arise as the compatibility condition of (2.4.2):

$$\dot{L}_n = A_{n+1} L_n - L_n A_n. \qquad (2.4.3)$$

The Lax pair for the AL model is [26]:

$$L_n^{(\mathrm{AL})} = \mathbb{N}_n \begin{pmatrix} \mathrm{e}^\lambda & \bar{\psi}_n \\ \psi_n & \mathrm{e}^{-\lambda} \end{pmatrix}, \qquad (2.4.4)$$

$$A_n^{(\mathrm{AL})} = \begin{pmatrix} \mathrm{e}^{2\lambda} + \psi_n \bar{\psi}_{n-1} - 2\bar{\psi}_n \psi_{n-1} - 1 & \mathrm{e}^\lambda \bar{\psi}_n - \mathrm{e}^{-\lambda} \bar{\psi}_{n-1} \\ \mathrm{e}^\lambda \psi_{n-1} - \mathrm{e}^{-\lambda} \psi_n & -\mathrm{e}^{-2\lambda} + 2\psi_n \bar{\psi}_{n-1} - \bar{\psi}_n \psi_{n-1} + 1 \end{pmatrix},$$

where $\mathbb{N}_n = (1 - |\psi_n|^2)$. In the same continuous limit as for the equations of motion, as well as with $\lambda \to \Delta\lambda$, this converts into the continuous Lax pair, (2.1.2).

2.4.2 Poisson Brackets

Like in the continuous case, we extract the Poisson brackets for the system from a particular combination of the spatial component of the Lax pair, L_n, and an associated r-matrix (which still needs to be a solution of the classical Yang-Baxter equation, (2.1.6)). The specific relation between the two is [17]:

$$\{L_{a,n}(\lambda), L_{b,m}(\mu)\} = [r_{ab}(\lambda - \mu), L_{a,n}(\lambda) L_{b,n}(\mu)] \, \delta_{nm}, \qquad (2.4.5)$$

---

[8]As with the auxiliary linear problem, whether the discrete or continuous Lax pair is being referred to when the generic name "Lax pair" is used will be clear from context.

where δ_{nm} is the Kronecker delta function[9]. Note that this has the same form as the Poisson bracket between the continuous monodromy matrices in (2.2.2), which shall still be true when we consider the discrete monodromy matrices, (2.5.2). In the continuum limit this reverts back to its continuous analogue (2.1.8), as detailed in [28, 29].

The r-matrix of the AL model takes a different form to the r-matrix for the NLS model. Instead of (2.1.7), we have:

$$r^{(\text{AL})}(\lambda) = \frac{-1}{e^\lambda - e^{-\lambda}} \begin{pmatrix} e^\lambda + e^{-\lambda} & 0 & 0 & 0 \\ 0 & 3e^{-\lambda} & -1 & 0 \\ 0 & -1 & 3e^\lambda & 0 \\ 0 & 0 & 0 & e^\lambda + e^{-\lambda} \end{pmatrix}. \tag{2.4.6}$$

In the continuum limit used for the Lax pair, $\lambda \to \Delta\lambda$, this reverts to the r-matrix used for the continuous NLS model, (2.1.7), up to some additive commuting factor.

Inserting the AL Lax matrix and r-matrix into (2.4.5), we find the Poisson brackets between the fields in this model:

$$\{\psi_n, \bar{\psi}_m\} = (1 - |\psi_n|^2)\delta_{nm}. \tag{2.4.7}$$

2.5 Discrete Periodic Boundary Conditions

2.5.1 Conserved Quantities

As with the continuous case, we start by defining the transport matrix T as a solution of the spatial component of the auxiliary linear problem, (2.4.2), in place of Ψ_n:

$$T(n, m; \lambda) = L_n(\lambda)...L_m(\lambda), \tag{2.5.1}$$

and for a system lying on the discrete interval $\{1, ..., N\}$, we call the transport

---

[9]Because the equal-space construction is only valid in the continuous case (as in the discrete case the dual picture would be considering some discrete space-evolution, and the idea of a Poisson structure (which the dual picture is built upon) does not exist in this setting), we do not need the S subscript on the Poisson bracket.

matrix that covers the whole interval the monodromy matrix $T(\lambda) = T(N, 1; \lambda)$. Due to both the commutator and the Poisson bracket obeying the Leibniz rule, it immediately follows that if L_n satisfies (2.4.5), then the monodromy matrix satisfies:

$$\{T_a(\lambda), T_b(\mu)\} = [r_{ab}(\lambda - \mu), T_a(\lambda)T_b(\mu)]. \tag{2.5.2}$$

For a periodic system ($L_{N+1} = L_1$) the trace of the monodromy matrix (called the transfer matrix), $\mathfrak{t}(\lambda) = \mathrm{tr}\{T(\lambda)\}$, Poisson commutes with itself for any choice of spectral parameters, $\{\mathfrak{t}(\lambda), \mathfrak{t}(\mu)\} = 0$. Therefore, if the transfer matrix $\mathfrak{t}(\lambda)$ is written as a power series in λ, the coefficients at different orders will also Poisson commute with one another:

$$\{\mathfrak{t}^{(k)}, \mathfrak{t}^{(j)}\} = 0, \tag{2.5.3}$$

where the coefficient of λ^k is labelled $\mathfrak{t}^{(k)}$.

If $L_n(\lambda)$ contains l different λ dependencies (for example, the AL Lax matrix (2.4.4) contains three, e^λ, $\mathrm{e}^{-\lambda}$, and λ^0), then the expansion of the monodromy matrix defined through the product (2.5.1) will contain $\big((l-1)N+1\big)$ terms (e.g. the expansion of the AL monodromy matrix will contain $(2N+1)$ terms, everything from $\mathrm{e}^{-N\lambda}$ through to $\mathrm{e}^{N\lambda}$). Then, because the Liouville-Arnold theorem defines a system with a finite number of degrees of freedom $2N$ as integrable if it has at least N Poisson commuting integrals of motion, and the expansion of the monodromy matrix provides $\big((l-1)N+1\big)$ Poisson commuting quantities, we see the first hints of integrability (as each site along the spatial axis will provide two degrees of freedom, and assuming that the Lax matrix has a non-trivialisable λ dependence).

By considering the time derivative of $\mathfrak{t}$, it is straightforward to see that it is constant:

$$\dot{\mathfrak{t}} = \mathrm{tr}\left\{\dot{T}\right\}$$

$$= \mathrm{tr}\left\{\sum_{n=1}^{N} L_N...L_{n+1}\dot{L}_n L_{n-1}...L_1\right\} \tag{2.5.4}$$

$$= \sum_{n=1}^{N} \mathrm{tr}\left\{L_N...L_{n+1}A_{n+1}L_n L_{n-1}...L_1 - L_N...L_{n+1}L_n A_n L_{n-1}...L_1\right\},$$

and by using the periodicity of the system under consideration to shift the summed index in one of the terms, this cancels out. Consequently, $\mathfrak{t}$ is a generator of Poisson commuting conserved quantities, providing the integrability of the system.

Unlike in the continuous case (and much for the better), the expansion of the transfer matrix can be explicitly calculated with relative ease, as it is merely a product of matrices, although we will once again be interested in the conserved quantities generated by the logarithm of the transfer matrix $\mathcal{G}(\lambda) = \ln(\mathfrak{t}(\lambda))$.

Example: Ablowitz-Ladik

The two ends of the expansion of the monodromy matrix built from the AL Lax matrix, (2.4.4), will be at order $\mathrm{e}^{-N\lambda}$ and $\mathrm{e}^{N\lambda}$:

$$T^{(\mathrm{AL})} = \mathbb{N}_N...\mathbb{N}_1 \left[\mathrm{e}^{-N\lambda} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{e}^{(1-N)\lambda} \begin{pmatrix} 0 & \bar{\psi}_N \\ \psi_1 & 0 \end{pmatrix} + \mathrm{e}^{(2-N)\lambda} T^{(\mathrm{AL},2-N)} + ... \right.$$

$$\left. + \mathrm{e}^{(N-2)\lambda} T^{(\mathrm{AL},N-2)} + \mathrm{e}^{(N-1)\lambda} \begin{pmatrix} 0 & \bar{\psi}_1 \\ \psi_N & 0 \end{pmatrix} + \mathrm{e}^{N\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], \qquad (2.5.5)$$

where:

$$T^{(\mathrm{AL},2-N)} = \begin{pmatrix} \bar{\psi}_N \psi_1 & 0 \\ 0 & \sum_n \psi_{n+1}\bar{\psi}_n - \bar{\psi}_N \psi_1 \end{pmatrix},$$

$$(2.5.6)$$

$$T^{(\mathrm{AL},N-2)} = \begin{pmatrix} \sum_n \bar{\psi}_{n+1}\psi_n - \psi_N \bar{\psi}_1 & 0 \\ 0 & \psi_N \bar{\psi}_1 \end{pmatrix}.$$

After taking the trace of this, the expansion of the transfer matrix is:

$$\mathfrak{t}^{(\mathrm{AL})} = \mathbb{N}_N...\mathbb{N}_1 \left[\mathrm{e}^{-N\lambda} + \mathrm{e}^{(2-N)\lambda} \sum_{n=1}^{N} \psi_{n+1}\bar{\psi}_n + ... + \mathrm{e}^{(N-2)\lambda} \sum_{n=1}^{N} \bar{\psi}_{n+1}\psi_n + \mathrm{e}^{N\lambda} \right].$$

$$(2.5.7)$$

In fact, due to the order λ^0 term in L_n being purely anti-diagonal, we can see that every second term in the expansion of $\mathfrak{t}(\lambda)$ will be 0 because of the trace. In order to expand $\mathcal{G}$ and find the conserved quantities that we are interested in, we need to choose which end of the series to focus on. In the limit as $\lambda \to +\infty$ the leading

term will be $e^{N\lambda}$, so that the expansion of the logarithm gives:

$$\mathcal{G}^{(\mathrm{AL})}(\lambda \to +\infty) = N\lambda + \sum_{n=1}^{N} \ln(\mathbb{N}_n) + e^{-2\lambda} \sum_{n=1}^{N} \bar{\psi}_{n+1}\psi_n$$

$$+ e^{-4\lambda} \sum_{n=1}^{N} \left(\bar{\psi}_{n+1}\psi_{n-1}(1 + |\psi_n|^2) + \frac{1}{2}\bar{\psi}_{n+1}^2\psi_n^2\right) + ...,$$

$$(2.5.8)$$

while in the limit as $\lambda \to -\infty$ the $e^{-N\lambda}$ term will dominate, giving:

$$\mathcal{G}^{(\mathrm{AL})}(\lambda \to -\infty) = -N\lambda + \sum_{n=1}^{N} \ln(\mathbb{N}_n) + e^{2\lambda} \sum_{n=1}^{N} \psi_{n+1}\bar{\psi}_n$$

$$+ e^{4\lambda} \sum_{n=1}^{N} \left(\psi_{n+1}\bar{\psi}_{n-1}(1 + |\psi_n|^2) + \frac{1}{2}\psi_{n+1}^2\bar{\psi}_n^2\right) + ...$$

$$(2.5.9)$$

In order to extract physical quantities from this we need to add the equivalent order results from both ends. Doing this gives the first three conserved quantities for the AL model as:

$$I_0^{(\mathrm{AL})} = 2\sum_{n=1}^{N} \ln(\mathbb{N}_n),$$

$$I_2^{(\mathrm{AL})} = \sum_{n=1}^{N} (\bar{\psi}_{n+1}\psi_n + \psi_{n+1}\bar{\psi}_n), \qquad (2.5.10)$$

$$I_4^{(\mathrm{AL})} = \sum_{n=1}^{N} \left((1 + |\psi_n|^2)(\bar{\psi}_{n+1}\psi_{n-1} + \psi_{n+1}\bar{\psi}_{n-1}) + \frac{1}{2}(\bar{\psi}_{n+1}^2\psi_n^2 + \psi_{n+1}^2\bar{\psi}_n^2)\right).$$

As can be seen these are all local quantities, with the non-locality of the order $e^{\pm 4\lambda}$ terms having been removed by the logarithm. The combination:

$$H^{(\mathrm{AL})} = I_2^{(\mathrm{AL})} + I_0^{(\mathrm{AL})} = \sum_{n=1}^{N} (\bar{\psi}_{n+1}\psi_n + \psi_{n+1}\bar{\psi}_n + 2\ln(\mathbb{N}_n)), \qquad (2.5.11)$$

can be recognised as the Hamiltonian for the AL model [26], such that using it in Hamilton's equation with the AL Poisson brackets, (2.4.7), returns the AL equations of motion, (2.4.1).

2.5.2 A_n-Matrices

For each of the conserved quantities generated through the method of the previous section, I_n, we can associate a corresponding time flow t_n, such that the conserved quantity generates the evolution of the fields along t_n. Each of the resulting evolution equations will also arise as the compatibility condition of a Lax pair. As the spatial component, L_n, supplies the hierarchy of conserved quantities, it is the A_n-matrix that will differ between each of these systems.

As in the continuous case, we can derive such a generator by considering Hamilton's equation with the entire generator of the conserved quantities, $\mathfrak{t}$, in place of the Hamiltonian, and equating this to the zero-curvature condition (but with the generator of the A_n-matrices, labelled $\mathbb{A}_n$):

$$
\partial_{\bar{t}} L_{b,n} = \mathbb{A}_{b,n+1}(\lambda, \mu) L_{b,n}(\lambda) - L_{b,n}(\lambda) \mathbb{A}_{b,n}(\lambda, \mu)
$$

$$
= \big\{ \ln \, \mathrm{tr}_a \{ T_a(\mu) \}, L_{bn}(\lambda) \big\}
$$

(2.5.12)

$$
= \mathfrak{t}^{-1}(\mu) \mathrm{tr}_a \left\{ L_{a,N}(\mu) ... L_{a,1}(\mu), L_{b,n}(\lambda) \right\}
$$

$$
= \mathfrak{t}^{-1} \mathrm{tr}_a \left\{ L_{a,N} ... L_{a,n+1} [r_{ab}(\mu - \lambda), L_{a,n}(\mu) L_{b,n}(\lambda)] L_{a,n-1} ... L_{a,1} \right\},
$$

where $\bar{t}$ is the universal time flow. By expanding the commutator in the last line of this, we can see that the generator for the A_n-matrices is:

$$
\mathbb{A}_n(\lambda, \mu) = \mathfrak{t}^{-1}(\mu) \mathrm{tr}_a \left\{ L_{a,N}(\mu) ... L_{a,n}(\mu) r_{ab}(\mu - \lambda) L_{a,n-1}(\mu) ... L_{a,1}(\mu) \right\}. \quad (2.5.13)
$$

If we expand this generator about powers of μ, the coefficient of μ^k will be the A_n-matrix that generates the same equations of motion as the conserved quantity appearing at order μ^k in the expansion of $\mathcal{G}(\mu)$. We could repeat this calculation, but starting from the generator $\mathfrak{t}(\lambda)$ instead. Then, the generator of the A-matrices would be:

$$
\mathbb{A}_n(\lambda, \mu) = \mathrm{tr}_a \left\{ L_{a,N}(\mu) ... L_{a,n}(\mu) r_{ab}(\mu - \lambda) L_{a,n-1}(\mu) ... L_{a,1}(\mu) \right\}. \quad (2.5.14)
$$

Example: Ablowitz-Ladik

Just as for the conserved quantities, we need to consider both limits of this model, $\lambda \to \pm\infty$. In the limit as $\lambda \to +\infty$, the first four A_n-matrices are:

$$\mathbb{A}_n^{(\text{AL},-0)} = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{A}_n^{(\text{AL},-1)} = \mathbb{A}_n^{(\text{AL},-3)} = 0,$$

$$\mathbb{A}_n^{(\text{AL},-2)} = -\begin{pmatrix} 2\mathrm{e}^{2\lambda} + 2\bar{\psi}_n\psi_{n-1} & -\mathrm{e}^{\lambda}\bar{\psi}_n \\ -\mathrm{e}^{\lambda}\psi_{n-1} & 3\mathrm{e}^{2\lambda} + \bar{\psi}_n\psi_{n-1} \end{pmatrix}.$$

(2.5.15)

In the limit as $\lambda \to -\infty$, the first four A_n-matrices are:

$$\mathbb{A}_n^{(\text{AL},0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{A}_n^{(\text{AL},1)} = \mathbb{A}_n^{(\text{AL},3)} = 0,$$

$$\mathbb{A}_n^{(\text{AL},2)} = \begin{pmatrix} 3\mathrm{e}^{-2\lambda} + \psi_n\bar{\psi}_{n-1} & -\mathrm{e}^{-\lambda}\bar{\psi}_{n-1} \\ -\mathrm{e}^{-\lambda}\psi_n & 2\mathrm{e}^{-2\lambda} + 2\psi_n\bar{\psi}_{n-1} \end{pmatrix},$$

(2.5.16)

so that, just as the Hamiltonian was built by combining the results from each end and adding the zeroth and second order terms, the temporal component of the AL Lax pair, (2.4.4), is built as:

$$A_n^{(\text{AL})} = (\mathbb{A}_n^{(\text{AL},2)} + \mathbb{A}_n^{(\text{AL},-2)}) + (\mathbb{A}_n^{(\text{AL},0)} + \mathbb{A}_n^{(\text{AL},-0)}), \qquad (2.5.17)$$

up to an overall factor of $3(\mathrm{e}^{-2\lambda} - \mathrm{e}^{2\lambda})\mathbb{I}$, which can be seen to leave the equations of motion invariant by looking at the zero-curvature condition, (2.4.3).

2.6 Discrete Open Boundary Conditions

2.6.1 Conserved Quantities

When we introduce non-periodic boundary conditions into models with discrete space, we need to introduce $K_{\pm}$-matrices that lie at the $n = N + 1$ (for K_+) and $n = 0$ (for K_-) boundaries. These are equivalent to the $K_{\pm}$-matrices from the

continuous discussion in Section 2.3, and as they were introduced at the level of the monodromy matrix through Sklyanin's double-row transfer matrix, (2.3.3), and the monodromy matrices in both the continuous and discrete cases satisfy the same algebras, (2.2.2) and (2.5.2) respectively, the $K_{\pm}$-matrices appear in the discrete transfer matrix in an identical manner:

$$\bar{\mathfrak{t}}(\lambda) = \operatorname{tr}\left\{K_+(\lambda)T(N,1;\lambda)K_-(\lambda)T^{-1}(N,1;-\lambda)\right\}, \qquad (2.6.1)$$

and need to satisfy (for non-dynamical boundary conditions, which are all we consider) the static classical reflection equation, (2.3.1). The conserved quantities that we are interested in appear as the coefficients in the expansion of $\bar{\mathcal{G}}(\lambda) = \ln\left(\bar{\mathfrak{t}}(\lambda)\right)$.

Example: Ablowitz-Ladik

The Ablowitz-Ladik model has previously been considered in the case of non-periodic boundary conditions in [30] and [31]. We re-derive their results for comparison against the open quantum Ablowitz-Ladik model discussed in Chapter 4.

The most general $K_{\pm}$-matrices that solve (2.3.1) with the choice of r-matrix for the AL model are:

$$K_{\pm}^{(\mathrm{AL})}(\lambda) = \begin{pmatrix} \alpha_{\pm}\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda} & 0 \\ 0 & \mathrm{e}^{\lambda} + \alpha_{\pm}\mathrm{e}^{-\lambda} \end{pmatrix}, \qquad (2.6.2)$$

where $\alpha_{\pm}$ are free parameters that lie at the two ends of the system, α_+ at $(N+1)$ and α_- at 0. Expanding the double-row transfer matrix as $\lambda \to +\infty$, the first two non-zero conserved quantities are:

$$\bar{I}_{-0}^{(\mathrm{AL})} = \ln\left(\alpha_+\right) + \ln\left(\alpha_-\right) - \sum_{n=1}^{N} \ln\left(\mathbb{N}_n\right),$$

$$(2.6.3)$$

$$\bar{I}_{-2}^{(\mathrm{AL})} = \frac{1}{\alpha_+}\mathbb{N}_N + \frac{1}{\alpha_-}\mathbb{N}_1 + \sum_{n=1}^{N-1}(\bar{\psi}_{n+1}\psi_n + \psi_{n+1}\bar{\psi}_n).$$

In the $\lambda \to -\infty$ limit, the first two non-trivial terms are identically:

$$\bar{I}_0^{(\mathrm{AL})} = \ln\left(\alpha_+\right) + \ln\left(\alpha_-\right) - \sum_{n=1}^{N} \ln\left(\mathbb{N}_n\right),$$

$$\bar{I}_2^{(\mathrm{AL})} = \frac{1}{\alpha_+}\mathbb{N}_N + \frac{1}{\alpha_-}\mathbb{N}_1 + \sum_{n=1}^{N-1}(\bar{\psi}_{n+1}\psi_n + \psi_{n+1}\bar{\psi}_n).$$

$$(2.6.4)$$

The closed Hamiltonian was found by taking the sum of the results in the two limits, however we can see here that the open Hamiltonian arises from either choice of limit (after removing the constant factors of $\alpha_\pm$ from $\bar{I}_0^{(\mathrm{AL})}$):

$$\bar{H}^{(\mathrm{AL})} = \bar{I}_{-2}^{(\mathrm{AL})} - 2\bar{I}_{-0}^{(\mathrm{AL})} = \bar{I}_2^{(\mathrm{AL})} - 2\bar{I}_0^{(\mathrm{AL})}$$

$$= \sum_{n=1}^{N-1}(\bar{\psi}_{n+1}\psi_n + \psi_{n+1}\bar{\psi}_n) + 2\sum_{n=1}^{N} \ln\left(\mathbb{N}_n\right) + \frac{1}{\alpha_+}\mathbb{N}_N + \frac{1}{\alpha_-}\mathbb{N}_1.$$

$$(2.6.5)$$

When we consider the Poisson brackets of this with the fields, we need to consider the bulk and boundary separately. In the bulk (that is, away from $n = 1, N$), we find the original equations of motion, (2.4.1):

$$\partial_t \psi_n = 2\psi_n - \mathbb{N}_n(\psi_{n+1} + \psi_{n-1}), \qquad \partial_t \bar{\psi}_n = -2\bar{\psi}_n + \mathbb{N}_n(\bar{\psi}_{n+1} + \bar{\psi}_{n-1}). \qquad (2.6.6)$$

At the $n = N$ boundary, we find:

$$\partial_t \psi_N = 2\psi_N - \mathbb{N}_N\left(\psi_{N-1} - \frac{\psi_N}{\alpha_+}\right), \qquad \partial_t \bar{\psi}_N = -2\bar{\psi}_N + \mathbb{N}_N\left(\bar{\psi}_{N-1} - \frac{\bar{\psi}_N}{\alpha_+}\right), \qquad (2.6.7)$$

whereas for the $n = 1$ boundary we get:

$$\partial_t \psi_1 = 2\psi_1 - \mathbb{N}_1\left(\psi_2 - \frac{\psi_1}{\alpha_-}\right), \qquad \partial_t \bar{\psi}_1 = -2\bar{\psi}_1 + \mathbb{N}_1\left(\bar{\psi}_2 - \frac{\bar{\psi}_1}{\alpha_-}\right). \qquad (2.6.8)$$

These agree with the results in [30] and [31].

2.6.2 A_n-Matrices

The final thing to do for the discrete models is to construct the A-matrices that provide the same boundary results as were found from the open Hamiltonians in

the previous subsection. To do this, we follow the derivation of the generator for the A-matrices with closed boundary conditions, (2.5.13), but replace the initial transfer matrix $\mathfrak{t}$ with the double-row version $\bar{\mathfrak{t}}$, from (2.6.1). The generator of the open A-matrices in the bulk is then:

$$\bar{\mathbb{A}}_{b,n} = \bar{\mathfrak{t}}^{-1}(\mu)\mathrm{tr}_a \left\{ K_{a,+}(\mu)T_a(N,n;\mu)r_{ab}(\mu - \lambda)T_a(n-1,1;\mu)K_{a,-}(\mu)T_a^{-1}(-\mu) \right.$$

$$\left. -K_{a,+}(\mu)T_a(\mu)K_{a,-}(\mu)T_a^{-1}(n-1,1;-\mu)r_{ab}(-\mu - \lambda)T_a^{-1}(N,n;-\mu) \right\},$$
$$(2.6.9)$$

where, in order to find the boundary results, we replace the appropriate monodromy matrices with the identity. By thinking of the zero-curvature condition, we can see that the $n = N$ boundary needs the matrix A_{N+1}, so the positive boundary is generated by:

$$\bar{\mathbb{A}}_{b,N+1} = \bar{\mathfrak{t}}^{-1}(\mu)\mathrm{tr}_a \left\{ K_{a,+}(\mu)r_{ab}(\mu - \lambda)T_a(\mu)K_{a,-}(\mu)T_a^{-1}(-\mu) \right.$$

$$(2.6.10)$$

$$\left. -K_{a,+}(\mu)T_a(\mu)K_{a,-}(\mu)T_a^{-1}(-\mu)r_{ab}(-\mu - \lambda) \right\}.$$

At the negative boundary $(n = 1)$ the generator is instead:

$$\bar{\mathbb{A}}_{b,1} = \bar{\mathfrak{t}}^{-1}(\mu)\mathrm{tr}_a \left\{ K_{a,+}(\mu)T_a(\mu)r_{ab}(\mu - \lambda)K_{a,-}(\mu)T_a^{-1}(-\mu) \right.$$

$$(2.6.11)$$

$$\left. -K_{a,+}(\mu)T_a(\mu)K_{a,-}(\mu)r_{ab}(-\mu - \lambda)T_a^{-1}(-\mu) \right\}.$$

When these are expanded about powers of μ, the coefficients of μ^k will give the same equations of motion (through the zero-curvature condition) as the open Hamiltonian (through Hamilton's equation) that appeared at order μ^k in the expansion of $\bar{\mathcal{G}}$.

Example: Ablowitz-Ladik

Expanding the bulk generator, (2.6.9), with the Ablowitz-Ladik r-matrix, (2.4.6), and $K_{\pm}$-matrices, (2.6.2), provides the open A-matrices for the AL model. Because the results found from the two separate limits are the same, we only need to consider the limit $\lambda \to \infty$, where the first two non-zero A-matrices (recalling that the odd-

powered terms are trivial) in the bulk are:

$$\mathbb{A}_n^{(\text{AL},0)} = -\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \tag{2.6.12}$$

$$\mathbb{A}_n^{(\text{AL},2)} = -(\text{e}^{2\lambda} + \text{e}^{-2\lambda}) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} \psi_n \bar{\psi}_{n-1} - 2\bar{\psi}_n \psi_{n-1} & \text{e}^{\lambda}\bar{\psi}_n - \text{e}^{-\lambda}\bar{\psi}_{n-1} \\ \text{e}^{\lambda}\psi_{n-1} - \text{e}^{-\lambda}\psi_n & 2\bar{\psi}_n \psi_{n-1} - \psi_n \bar{\psi}_{n-1} \end{pmatrix},$$

while at the positive boundary we have:

$$\mathbb{A}_{N+1}^{(\text{AL},0)} = -\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \tag{2.6.13}$$

$$\mathbb{A}_{N+1}^{(\text{AL},2)} = -(\text{e}^{2\lambda} + \text{e}^{-2\lambda}) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{\alpha_+}|\psi_N|^2 & -(\frac{1}{\alpha_+}\text{e}^{\lambda} + \text{e}^{-\lambda})\bar{\psi}_N \\ (\text{e}^{\lambda} + \frac{1}{\alpha_+}\text{e}^{-\lambda})\psi_N & \frac{-1}{\alpha_+}|\psi_N|^2 \end{pmatrix},$$

and at the negative boundary:

$$\mathbb{A}_1^{(\text{AL},0)} = -\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \tag{2.6.14}$$

$$\mathbb{A}_1^{(\text{AL},2)} = -(\text{e}^{2\lambda} + \text{e}^{-2\lambda}) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{\alpha_-}|\psi_1|^2 & (\text{e}^{\lambda} + \frac{1}{\alpha_-}\text{e}^{-\lambda})\bar{\psi}_1 \\ -(\frac{1}{\alpha_-}\text{e}^{\lambda} + \text{e}^{-\lambda})\psi_1 & \frac{-1}{\alpha_-}|\psi_1|^2 \end{pmatrix}.$$

Finally, in parallel to how the open Hamiltonian was constructed, we find the time halves of the AL Lax pair in the presence of open boundary conditions by considering:

$$A_1^{(\text{AL})} = \mathbb{A}_1^{(\text{AL},2)} - 2\mathbb{A}_1^{(\text{AL},0)} + (3\text{e}^{2\lambda} + 2\text{e}^{-2\lambda} + 5)\mathbb{I},$$

$$A_n^{(\text{AL})} = \mathbb{A}_n^{(\text{AL},2)} - 2\mathbb{A}_n^{(\text{AL},0)} + (3\text{e}^{2\lambda} + 2\text{e}^{-2\lambda} + 5)\mathbb{I}, \tag{2.6.15}$$

$$A_{N+1}^{(\text{AL})} = \mathbb{A}_{N+1}^{(\text{AL},2)} - 2\mathbb{A}_{N+1}^{(\text{AL},0)} + (3\text{e}^{2\lambda} + 2\text{e}^{-2\lambda} + 5)\mathbb{I},$$

where the extra constant commuting factors are merely added to make the expressions simpler.

Chapter 3

Dual Hierarchies

By considering the equal prominence of the space and time coordinates in the Lagrangian picture of a (1+1)-dimensional system, a dual Hamiltonian formulation of the non-linear Schrödinger model was constructed in [13], which had equal-space Poisson brackets (in place of the equal-time Poisson brackets) and dual integrals of motion that are conserved with respect to space-evolution rather than time-evolution. We focus here on the Lax pair construction rather than the Lagrangian picture emphasised in previous works.

In this chapter, we repeat the dual construction of the non-linear Schrödinger model (from [13]) and build the dual construction of the isotropic Landau-Lifshitz model in the language of Lax pairs. It follows mostly in parallel with Chapter 2, with the only divergences being where we emphasise important differences between the two pictures, such as in the limiting procedure of the exponential in the case of open boundary conditions, and where we digress to give an example of how this dual picture can be used to find integrable systems depending non-trivially on additional fields.

Section 3.4 considers the introduction of time-like boundary conditions. This idea was first introduced in [11], where it was applied to the NLS model, and was later applied to the HM model in [10]. The chapter is built out of these two papers.

3.1 Poisson Brackets

The first step in this dual construction is defining the equal-space Poisson brackets (3.1.8) through the use of the r-matrix and an analogue of the linear algebraic relation. However, as the hierarchy will now describe a series of commuting space flows, the ψ' and $\bar{\psi}'$ in the NLS V-matrix (2.1.2) and the S'_σ in the HM V-matrix (2.1.3) will all be derivatives with respect to a specific space flow. For the NLS model this will be the 2nd order flow, x_2, while for the HM model this will be the 0th order flow, x_0. These choices of flow will be justified later. Consequently, to prevent later confusion, we define these as some new fields, ϕ and $\bar{\phi}$ for the NLS model and Σ_σ for the HM model.

These new fields will be identified with their appropriate derivatives when we consider the equations of motion for the appropriate dual Hamiltonian or V-matrix (namely, the 2nd order results for the NLS model and the 0th order for the HM model). Specifically, these results can be seen in (3.2.14) for the NLS model and (3.2.17) for the HM model. When we work with the equations of motion found from a different space flow (as we will in Section 3.3), these new fields will be shown to act as wholly independent fields, with their own distinct space-evolution.

With these new fields, the NLS V-matrix that we consider is:

$$V^{(\text{NLS})} = \begin{pmatrix} \frac{\lambda^2}{2} - |\psi|^2 & \lambda\bar{\psi} + \bar{\phi} \\ \lambda\psi - \phi & -\frac{\lambda^2}{2} + |\psi|^2 \end{pmatrix}, \tag{3.1.1}$$

and the HM V-matrix is:

$$V^{(\text{HM})} = \frac{1}{2\lambda^2}S - \frac{1}{2c^2\lambda}\Sigma S, \tag{3.1.2}$$

with:

$$\Sigma = \begin{pmatrix} \Sigma_z & \Sigma_- \\ \Sigma_+ & -\Sigma_z \end{pmatrix}. \tag{3.1.3}$$

In the standard picture the Poisson brackets are related to the U- and r-matrices through (2.1.8), so we would expect a similar relation between the equal-space Poisson brackets and the V- and r-matrices. Indeed, in the original works [12, 13] the

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dual NLS Poisson brackets were extracted from the Lagrangian picture and then written in terms of the standard Yangian r-matrix through:

$$\{V_a(t_1, \lambda), V_b(t_2, \mu)\}_T = [r_{ab}(\lambda - \mu), V_a(t_1, \lambda) + V_b(t_2, \mu)] \, \delta(t_1 - t_2). \qquad (3.1.4)$$

By describing the NLS space-flow as a particular choice of one time-flow in an underlying $(0 + 1)$-dimensional system, a similar r-matrix structure was provided for the whole NLS hierarchy in [32]. As that process built off of the underlying $\mathfrak{su}_2$ form of the Lax matrices, it will apply to the HM model as well, allowing us to consider the dual construction of the HM model in terms of (3.1.4).

For the NLS model, inserting both the modified V-matrix, (3.1.1), and the same r-matrix as for the standard construction, (2.1.7), the dual Poisson brackets between the four fields are:

$$\{\psi(t_1), \bar{\psi}(t_2)\}_T = \{\phi(t_1), \bar{\phi}(t_2)\}_T = 0,$$

$$\{\psi(t_1), \phi(t_2)\}_T = \{\bar{\psi}(t_1), \bar{\phi}(t_2)\}_T = 0, \qquad (3.1.5)$$

$$\{\psi(t_1), \bar{\phi}(t_2)\}_T = \{\bar{\psi}(t_1), \phi(t_2)\}_T = \delta(t_1 - t_2).$$

These take the form of two conjugate pairs, $(\psi, \bar{\phi})$ and $(\bar{\psi}, \phi)$.

When we insert the modified HM V-matrix, (3.1.2), and the original HM r-matrix, (2.1.7), into the expression for the dual Poisson brackets, (3.1.4), the dual Poisson brackets for the HM model are:

$$\{S_\pm(t_1), S_z(t_2)\}_T = \{S_+(t_1), S_-(t_2)\}_T = 0,$$

$$\{S_\pm(t_1), \Sigma_z(t_2)\}_T = \{S_z(t_1), \Sigma_\pm(t_2)\}_T = S_\pm S_z \, \delta(t_1 - t_2),$$

$$\{S_z(t_1), \Sigma_z(t_2)\}_T = -S_+ S_- \, \delta(t_1 - t_2),$$

$$\{S_\pm(t_1), \Sigma_\pm(t_2)\}_T = S_\pm^2 \, \delta(t_1 - t_2), \qquad (3.1.6)$$

$$\{S_\pm(t_1), \Sigma_\mp(t_2)\}_T = -(2S_z^2 + S_+ S_-) \, \delta(t_1 - t_2),$$

$$\{\Sigma_{\sigma_1}(t_1), \Sigma_{\sigma_2}(t_2)\}_T = (S_{\sigma_1} \Sigma_{\sigma_2} - \Sigma_{\sigma_1} S_{\sigma_2}) \, \delta(t_1 - t_2).$$

As well as sharing the Casimir element $c^2 = S_z^2 + S_+ S_-$ with the original model, these brackets have an additional commuting quantity:

$$\tilde{c}^2 = 2S_z \Sigma_z + S_+ \Sigma_- + S_- \Sigma_+, \tag{3.1.7}$$

where, in reference to how $\Sigma_\sigma = \partial_{x_0} S_\sigma$ in the HM model, we choose to set $\tilde{c} = 0$. Consequently, when the HM model is considered and we can write the Σ_σ directly as the derivatives of the S_σ, (3.1.7) becomes redundant as it is merely the derivative of the original Casimir, (2.1.11). At any other level of the hierarchy however, we cannot directly relate the Σ_σ and the S_σ, so the two Casimirs are distinct.

Introducing the fields Σ_x, Σ_y, and Σ_z in analogy to S_x, S_y, and S_z, these Poisson brackets can be written more compactly by using the indices $i, j \in \{x, y, z\}$:

$$\{S_i(t_1), S_j(t_2)\}_T = 0,$$

$$\{S_i(t_1), \Sigma_j(t_2)\}_T = \left(S_i S_j - c^2 \delta_{ij}\right) \delta(t_1 - t_2), \tag{3.1.8}$$

$$\{\Sigma_i(t_1), \Sigma_j(t_2)\}_T = \left(S_i \Sigma_j - S_j \Sigma_i\right) \delta(t_1 - t_2),$$

where the two Casimir elements are now:

$$c^2 = S_x^2 + S_y^2 + S_z^2, \qquad 0 = S_x \Sigma_x + S_y \Sigma_y + S_z \Sigma_z. \tag{3.1.9}$$

Finally, by defining the quantities:

$$\psi_1 = S_x^2, \qquad \phi_1 = \frac{1}{2c^2}\left(\frac{\Sigma_z}{S_z} - \frac{\Sigma_x}{S_x}\right),$$

$$\psi_2 = S_y^2, \qquad \phi_2 = \frac{1}{2c^2}\left(\frac{\Sigma_z}{S_z} - \frac{\Sigma_y}{S_y}\right), \tag{3.1.10}$$

the above Poisson brackets can be written in terms of two conjugate pairs (ψ_1, ϕ_1) and (ψ_2, ϕ_2), where we use the two Casimir elements to discount two of the fields:

$$\{\psi_1(t_1), \psi_2(t_2)\}_T = \{\phi_1(t_1), \phi_2(t_2)\}_T = 0,$$

$$\{\psi_i(t_1), \phi_j(t_2)\}_T = \delta_{ij}\delta(t_1 - t_2). \tag{3.1.11}$$

Due however to the complicated form of the ϕ_i, we continue to work with the S_σ and Σ_σ in what follows.

Further discussions on the Poisson structures arising here, as well as some more general results, are presented in Appendix A.

3.2   Periodic Boundary Conditions

In both this section and the next (where open boundary conditions are considered), we consider a system that lies on the interval $[-\tau, \tau]$, for some $\tau > 0$. The periodic boundary conditions in this setting are then $\psi(\tau) = \psi(-\tau)$, $\phi(\tau) = \phi(-\tau)$, and their complex conjugates for the NLS model, and $S_\sigma(\tau) = S_\sigma(-\tau)$ and $\Sigma_\sigma(\tau) = \Sigma_\sigma(-\tau)$ for the HM model.

3.2.1   Conserved Quantities

The construction of the dual model follows in parallel with Section 2.2. The first object constructed is therefore the equal-space monodromy matrix, T_T, which is a solution in place of Ψ to the temporal half of the auxiliary linear problem, (2.1.1). This is diagonalised (by analogy with the standard picture discussed in Chapter 2) through the use of a diagonal matrix Z_T and an anti-diagonal matrix W_T:

$$
\begin{aligned}
T_T(t_1, t_2; \lambda) &= \mathrm{P} \exp \int_{t_2}^{t_1} V(\xi) \mathrm{d}\xi \\
&= \big(\mathbb{I} + W_T(t_1; \lambda)\big) \mathrm{e}^{Z_T(t_1, t_2; \lambda)} \big(\mathbb{I} + W_T(t_2; \lambda)\big)^{-1}.
\end{aligned}
\tag{3.2.1}
$$

Because we have chosen that the V-matrices satisfy a linear algebraic relation of the form (3.1.4), the full equal-space monodromy matrix $T_T(\lambda) = T_T(\tau, -\tau; \lambda)$ will satisfy a quadratic algebraic relation analogous to (2.2.2):

$$
\{T_{T,a}(\lambda), T_{T,b}(\mu)\}_T = [r_{ab}(\lambda - \mu), T_{T,a}(\lambda)T_{T,b}(\mu)].
\tag{3.2.2}
$$

Taking the trace of the equal-space monodromy matrix we get the equal-space transfer matrix, $\mathfrak{t}_T$:

$$\mathfrak{t}_T(\lambda) = \mathrm{tr}\left\{T_T(\lambda)\right\}$$

$$= e^{Z_{11,T}(\lambda)} + e^{Z_{22,T}(\lambda)}, \tag{3.2.3}$$

which, by virtue of the equal-space monodromy matrix satisfying the quadratic relation (3.2.2), Poisson commute for different spectral parameters:

$$\{\mathfrak{t}_T(\lambda), \mathfrak{t}_T(\mu)\}_T = 0. \tag{3.2.4}$$

Finally, as these two series Poisson commute, so will each pair of the coefficients $\mathfrak{t}_T^{(k)}$. Therefore, if we take the logarithm of these, $\mathcal{G}_T(\lambda) = \ln\left(\mathfrak{t}_T(\lambda)\right)$, we have that the coefficients in the series expansion of $\mathcal{G}_T(\lambda)$ Poisson commute with one another:

$$\{\mathcal{G}_T^{(k)}, \mathcal{G}_T^{(j)}\}_T = 0. \tag{3.2.5}$$

As in Section 2.2, in order to expand $\mathcal{G}_T$ we need to consider the leading order contribution in each of $Z_{11,T}$ and $Z_{22,T}$. Consequently, if we insert the diagonalisation of T_T into the temporal half of the auxiliary linear problem, (2.1.1), then we find relations for the W_T and Z_T:

$$0 = \dot{W}_T + [W_T, V_D] + W_T V_A W_T - V_A,$$

$$\dot{Z}_T = V_D + V_A W_T, \tag{3.2.6}$$

where now V_D and V_A are the diagonal and anti-diagonal components of the V-matrix, respectively. Expanding W_T and Z_T in powers of λ as[1]:

$$W_T(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k} W_T^{(k)}, \qquad Z_T(\lambda) = \sum_{k=-2}^{\infty} \lambda^{-k} Z_T^{(k)}, \tag{3.2.7}$$

---

[1]Note that due to the underlying V-matrix having a higher dependence on λ (λ^2 for NLS and λ^{-2} for HM) than the highest dependence in the corresponding U-matrix (λ for NLS and λ^{-1} for HM), the Z_T series will need to start at $k = -2$ instead of $k = -1$.

where again we switch from the NLS model to the HM model by replacing $\lambda^{-1} \to \lambda$ due to the different limits being considered. We can then recursively solve (3.2.6). Solving the first few orders of these, the first six Z_T-matrices for the dual construction of the NLS model are:

$$Z_T^{(\mathrm{NLS},-2)} = \tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Z_T^{(\mathrm{NLS},-1)} = Z_T^{(\mathrm{NLS},0)} = 0,$$

$$Z_T^{(\mathrm{NLS},1)} = \int_{-\tau}^{\tau} (\psi\bar{\phi} - \phi\bar{\psi})\mathrm{d}t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Z_T^{(\mathrm{NLS},2)} = \int_{-\tau}^{\tau} \left[(|\psi|^4 - |\phi|^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \dot{\psi}\bar{\phi} & 0 \\ 0 & \psi\dot{\bar{\psi}} \end{pmatrix} \right] \mathrm{d}t,$$

$$Z_T^{(\mathrm{NLS},3)} = \int_{-\tau}^{\tau} \begin{pmatrix} \bar{\psi}\dot{\phi} - \dot{\psi}\bar{\phi} & 0 \\ 0 & \dot{\bar{\psi}}\phi - \psi\dot{\bar{\phi}} \end{pmatrix} \mathrm{d}t,$$

(3.2.8)

while for the HM model, the first three Z_T-matrices are:

$$Z_T^{(\mathrm{HM},-2)} = c\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Z_T^{(\mathrm{HM},-1)} = 0,$$

(3.2.9)

$$Z_T^{(\mathrm{HM},0)} = \frac{1}{2c} \int_{-\tau}^{\tau} \left[\dot{S}_z \mathbb{I} + (c - S_z) \begin{pmatrix} \frac{\dot{S}_-}{S_-} & 0 \\ 0 & -\frac{\dot{S}_+}{S_+} \end{pmatrix} - \frac{1}{2c^2}(\Sigma_+\Sigma_- + \Sigma_z^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \mathrm{d}t.$$

In both of these sequences, the $Z_T^{(-2)}$ has positive $Z_{T,11}^{(-2)}$ and negative $Z_{T,22}^{(-2)}$, so that in the considered limits of λ the $\mathrm{e}^{Z_{11,T}}$ terms dominate over the $\mathrm{e}^{Z_{22,T}}$ in (3.2.3). This allows us to simply expand the generator of the conserved quantities as:

$$\mathcal{G}_T = Z_{11,T} + ...,$$

(3.2.10)

just like in the equal-time picture (2.2.13). For the NLS model, the first six dual conserved quantities are therefore:

$$\mathcal{G}_T^{(\text{NLS},-2)} = \tau, \qquad\qquad \mathcal{G}_T^{(\text{NLS},-1)} = \mathcal{G}_T^{(\text{NLS},0)} = 0,$$

$$\mathcal{G}_T^{(\text{NLS},1)} = \int_{-\tau}^{\tau} \left(\psi\bar{\phi} - \phi\bar{\psi}\right) \mathrm{d}t,$$

$$\mathcal{G}_T^{(\text{NLS},2)} = \int_{-\tau}^{\tau} \left(|\psi|^4 - |\phi|^2 - \dot{\psi}\bar{\psi}\right) \mathrm{d}t, \qquad (3.2.11)$$

$$\mathcal{G}_T^{(\text{NLS},3)} = \int_{-\tau}^{\tau} \left(\bar{\psi}\dot{\phi} - \dot{\psi}\bar{\phi}\right) \mathrm{d}t,$$

the second of which, $\mathcal{G}_T^{(\text{NLS},2)}$, we shall call the dual Hamiltonian for the NLS model:

$$H_T^{(\text{NLS})} = \int_{-\tau}^{\tau} \left(|\psi|^4 - |\phi|^2 - \dot{\psi}\bar{\psi}\right) \mathrm{d}t. \qquad (3.2.12)$$

Our reason for doing this is that if we combine this with the dual Poisson brackets in (3.1.5) in a space-evolution analogue of Hamilton's equation:

$$\mathfrak{f}'(x,t) = \{H_T, \mathfrak{f}(x,t)\}_T, \qquad (3.2.13)$$

then we find four space-evolution equations:

$$\psi' = \phi, \qquad\qquad \bar{\psi}' = \phi',$$

$$\phi' = 2\psi|\psi|^2 - \dot{\psi}, \qquad\qquad \bar{\phi}' = 2\bar{\psi}|\psi|^2 + \dot{\bar{\psi}}. \qquad (3.2.14)$$

These can be recognised as the original equations of motion for the NLS model, (1.1.1). As expected, the identification of the new fields ϕ and $\bar{\phi}$ with the spatial derivatives of the original fields ψ and $\bar{\psi}$ appears as part of the dual equations of motion.

For the HM model, the first three conserved quantities are instead:

$$\mathcal{G}_T^{(\text{HM},-2)} = c\tau, \qquad\qquad \mathcal{G}_T^{(\text{HM},-1)} = 0,$$

$$\mathcal{G}_T^{(\text{HM},0)} = \frac{1}{2c} \int_{-\tau}^{\tau} \left(\dot{S}_z + (c - S_z)\frac{\dot{S}_-}{S_-} - \frac{1}{2c^2}(\Sigma_+\Sigma_- + \Sigma_z^2)\right) \mathrm{d}t. \qquad (3.2.15)$$

Focussing on the third of these, if we use the periodic boundary conditions to remove any total derivatives and multiply by a factor of $-2c$, $\mathcal{G}_T^{(\mathrm{HM},0)}$ reduces to the dual Hamiltonian for the HM model:

$$H_T^{(\mathrm{HM})} = \frac{1}{2} \int_{-L}^{L} \left(\frac{\dot{S}_+ S_- - S_+ \dot{S}_-}{c + S_z} + \frac{1}{c^2}(\Sigma_+ \Sigma_- + \Sigma_z^2) \right) \mathrm{d}t. \tag{3.2.16}$$

Using this in the space-evolutive Hamilton's equation, the space-evolution equations are:

$$S_\sigma' = \Sigma_\sigma,$$

$$\Sigma_\pm' = \pm(S_\pm \dot{S}_z - \dot{S}_\pm S_z) - \frac{1}{c^2} S_\pm (\Sigma_+ \Sigma_- + \Sigma_z^2), \tag{3.2.17}$$

$$\Sigma_z' = \frac{1}{2}(\dot{S}_+ S_- - S_+ \dot{S}_-) - \frac{1}{c^2} S_z (\Sigma_+ \Sigma_- + \Sigma_z^2),$$

which are equivalent to the original equations of motion for the HM model, (1.1.3). Again, the identification of the new fields with the derivatives of the old arises naturally from the space-evolution equations.

3.2.2 *U*-Matrices

Using the equal space Poisson brackets and the tower of equal-space conserved quantities, we can generate a whole hierarchy of space-evolution equations associated to distinct systems. Consequently, we will also be interested in generating Lax pairs for each of these systems. By following the derivation of (2.2.20) and (2.2.22), we can derive a generator $\mathbb{U}$ for the tower of U-matrices that partner with the underlying V-matrix of each of our systems, (3.1.1) for NLS and (3.1.2) for HM, which can be generally written as:

$$\mathbb{U}_b(t; \lambda, \mu) = \mathfrak{t}_T^{-1}(\mu)\mathrm{tr}_a \left\{ T_{T,a}(\tau, t; \mu) r_{ab}(\mu - \lambda) T_{T,a}(t, -\tau; \mu) \right\}, \tag{3.2.18}$$

or by using the known results and properties for the r-matrix, as well as the diagonalisation of T_T, this can be reduced to an expression that lies only in one vector space:

$$\mathbb{U}^{(\mathrm{NLS})}(t; \lambda, \mu) = 2\mathbb{U}^{(\mathrm{HM})}(x; \lambda, \mu) = \frac{1}{\mu - \lambda} \left(\mathbb{I} + W_T(t; \mu) \right) e_{11} \left(\mathbb{I} + W_T(t; \mu) \right)^{-1}. \tag{3.2.19}$$

For the NLS model, when we expand this generator about $\mu \to \infty$, the first four terms are:

$$\mathbb{U}^{(\text{NLS},1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbb{U}^{(\text{NLS},2)} = \begin{pmatrix} \lambda & \bar{\psi} \\ \psi & 0 \end{pmatrix},$$

$$\mathbb{U}^{(\text{NLS},3)} = \begin{pmatrix} \lambda^2 - |\psi|^2 & \lambda\bar{\psi} + \bar{\phi} \\ \lambda\psi - \phi & |\psi|^2 \end{pmatrix},$$

$$\mathbb{U}^{(\text{NLS},4)} = \begin{pmatrix} \lambda^3 - \lambda|\psi|^2 + (\bar{\psi}\phi - \psi\bar{\phi}) & \lambda^2\bar{\psi} + \lambda\bar{\phi} + \dot{\bar{\psi}} \\ \lambda^2\psi - \lambda\phi - \dot{\psi} & \lambda|\psi|^2 - (\bar{\psi}\phi - \psi\bar{\phi}) \end{pmatrix}.$$

(3.2.20)

Up to the addition of a factor of $-\frac{\lambda}{2}\mathbb{I}$, the second of these can be recognised as the spatial component of the NLS Lax pair, (2.1.2):

$$U^{(\text{NLS})} = \mathbb{U}^{(\text{NLS},2)} - \frac{\lambda}{2}\mathbb{I}. \tag{3.2.21}$$

Considering instead the HM model, the first three terms in the expansion of the generator about $\mu \to 0^+$ are:

$$\mathbb{U}^{(\text{HM},0)} = \frac{-1}{4\lambda}\mathbb{I} - \frac{1}{4c\lambda}S,$$

$$\mathbb{U}^{(\text{HM},1)} = \frac{-1}{4\lambda^2}\mathbb{I} - \frac{1}{4c\lambda^2}S + \frac{1}{4c^3\lambda}\Sigma S,$$

(3.2.22)

$$\mathbb{U}^{(\text{HM},2)} = \frac{-1}{4\lambda^3}\mathbb{I} - \frac{1}{4c\lambda^3}S + \frac{1}{4c^3\lambda^2}\Sigma S + \frac{1}{4c^3\lambda}\dot{S}S - \frac{1}{8c^5\lambda}\Sigma^2 S.$$

If we remove the constant factor from the first of these and multiply by a factor of $-2c$, $\mathbb{U}^{(\text{HM},0)}$ can then be identified with the spatial component of the HM Lax pair (2.1.3):

$$U^{(\text{HM})} = -2c(\mathbb{U}^{(\text{HM},0)} + \frac{1}{4\lambda}\mathbb{I}). \tag{3.2.23}$$

That these U-matrices agree with the known spatial components of the Lax pairs guarantees that the equations of motion from the zero-curvature condition in this

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dual picture match the equations of motion arising from the zero-curvature condition in the original picture, (1.1.1) for the NLS model and (1.1.3) for the HM model.

3.3 Higher Order Systems

The identification of the ϕ and Σ_σ with the derivatives of the ψ and S_σ, respectively, appears as part of the equations of motion for the NLS and the HM models. If we instead consider a different system in their hierarchies, these identifications will not necessarily occur. We demonstrate this here for each of the two systems under study.

Non-Linear Schrödinger

For the NLS model, we consider the system at order λ^4 in the dual hierarchy. This has Lax pair $(\mathbb{U}^{(\text{NLS},4)}, V)$, where V is the usual NLS V-matrix taken from (3.1.1) and $\mathbb{U}^{(\text{NLS},4)}$ is read from (3.2.20). If we insert this Lax pair into the zero-curvature condition, we find the space-evolution of this new system:

$$\psi' = -\dot{\phi} - 2\psi(\bar{\psi}\phi - \psi\bar{\phi}),$$

$$\bar{\psi}' = \dot{\bar{\phi}} + 2\bar{\psi}(\bar{\psi}\phi - \psi\bar{\phi}),$$

$$\phi' = \ddot{\psi} - 2\dot{\psi}|\psi|^2 - 2\phi(\bar{\psi}\phi - \psi\bar{\phi}),$$

$$\bar{\phi}' = \ddot{\bar{\psi}} + 2\dot{\bar{\psi}}|\psi|^2 + 2\bar{\phi}(\bar{\psi}\phi - \psi\bar{\phi}).$$

(3.3.1)

As stated above, these equations do not identify $\phi = \psi'$ and $\bar{\phi} = \bar{\psi}'$, so describe the space-evolution for four distinct fields.

Isotropic Landau-Lifshitz

For the HM model, we consider the system at order λ^2 in the dual hierarchy. This has Lax pair (U_2, V), where V is the HM V-matrix taken from (3.1.2) and we define:

$$U_2 = -2c(\mathbb{U}^{(\text{HM},2)} + \frac{1}{4\lambda^3}\mathbb{I})$$

$$= \frac{1}{2\lambda^3}S - \frac{1}{2c^2\lambda^2}\Sigma S - \frac{1}{2c^2\lambda}\dot{S}S + \frac{1}{4c^4\lambda}\Sigma^2 S.$$

(3.3.2)

Inserting this Lax pair into the zero-curvature condition, we find the space-evolution equations for this new system. The space-evolution of the three original fields, $S_\pm$ and S_z, are:

$$S'_+ = \frac{1}{c^2}(S_+\dot{\Sigma}_z - S_z\dot{\Sigma}_+) + \frac{1}{2c^4}\Sigma_+(\Sigma_z^2 + \Sigma_+\Sigma_-),$$

$$S'_- = \frac{1}{c^2}(S_z\dot{\Sigma}_- - S_-\dot{\Sigma}_z) + \frac{1}{2c^4}\Sigma_-(\Sigma_z^2 + \Sigma_+\Sigma_-), \tag{3.3.3}$$

$$S'_z = \frac{1}{2c^2}(S_-\dot{\Sigma}_+ - S_+\dot{\Sigma}_-) + \frac{1}{2c^4}\Sigma_z(\Sigma_z^2 + \Sigma_+\Sigma_-),$$

while the space-evolution of the three fields $\Sigma_\pm$ and Σ_z are:

$$\Sigma'_+ = \frac{1}{c^2}(\Sigma_+\dot{\Sigma}_z - \dot{\Sigma}_+\Sigma_z) + \ddot{S}_+ + \frac{1}{2c^4}\Sigma_+\Sigma_z(\dot{S}_+S_- - S_+\dot{S}_-)$$

$$+ \frac{1}{2c^4}\left(\Sigma_z^2(\dot{S}_+S_z - S_+\dot{S}_z) + \Sigma_+^2(\dot{S}_-S_z - S_-\dot{S}_z)\right)$$

$$+ S_+\left(\frac{1}{c^2}\left((\dot{S}_z)^2 + \dot{S}_+\dot{S}_-\right) - \frac{1}{2c^6}\left(\Sigma_z^2 + \Sigma_+\Sigma_-\right)^2\right),$$

$$\Sigma'_- = \frac{1}{c^2}(\dot{\Sigma}_-\Sigma_z - \Sigma_-\dot{\Sigma}_z) + \ddot{S}_- + \frac{1}{2c^4}\Sigma_-\Sigma_z(\dot{S}_+S_- - S_+\dot{S}_-)$$

$$+ \frac{1}{2c^4}\left(\Sigma_z^2(S_-\dot{S}_z - \dot{S}_-S_z) + \Sigma_-^2(S_+\dot{S}_z - \dot{S}_+S_z)\right) \tag{3.3.4}$$

$$+ S_-\left(\frac{1}{c^2}\left((\dot{S}_z)^2 + \dot{S}_+\dot{S}_-\right) - \frac{1}{2c^6}\left(\Sigma_z^2 + \Sigma_+\Sigma_-\right)^2\right),$$

$$\Sigma'_z = \frac{1}{2c^2}(\dot{\Sigma}_+\Sigma_- - \Sigma_+\dot{\Sigma}_-) + \ddot{S}_z + \frac{1}{4c^4}(\Sigma_z^2 - \Sigma_+\Sigma_-)(\dot{S}_+S_- - S_+\dot{S}_-)$$

$$+ \frac{1}{2c^4}\left(\Sigma_-\Sigma_z(S_+\dot{S}_z - \dot{S}_+S_z) + \Sigma_+\Sigma_z(\dot{S}_-S_z - S_-\dot{S}_z)\right)$$

$$+ S_z\left(\frac{1}{c^2}\left((\dot{S}_z)^2 + \dot{S}_+\dot{S}_-\right) - \frac{1}{2c^6}\left(\Sigma_z^2 + \Sigma_+\Sigma_-\right)^2\right).$$

These can be written more compactly in terms of the vectors $\vec{S} = (S_x, S_y, S_z)^T$ and $\vec{\Sigma} = (\Sigma_x, \Sigma_y, \Sigma_z)^T$ as:

$$\vec{S}' = \frac{i}{c^2}(\vec{S} \times \dot{\vec{\Sigma}}) + \frac{1}{2c^4}|\vec{\Sigma}|^2\vec{\Sigma}, \tag{3.3.5}$$

$$\vec{\Sigma}' = \frac{i}{c^2}(\vec{\Sigma} \times \dot{\vec{\Sigma}}) - \frac{i}{2c^4}|\vec{\Sigma}|^2(\vec{S} \times \dot{\vec{S}}) + \ddot{\vec{S}} + \vec{S}\left(\frac{1}{c^2}|\dot{\vec{S}}|^2 - \frac{1}{2c^6}|\vec{\Sigma}|^4\right) + \frac{i}{c^4}\vec{\Sigma}(\vec{\Sigma} \cdot (\vec{S} \times \dot{\vec{S}})).$$

3.4 Open Boundary Conditions

3.4.1 Conserved Quantities

Finally, we consider the effect of introducing reflective boundary conditions to the time-axis. This idea was introduced in [11], where it was applied to the NLS model. Due to the r-matrix structure for the dual model, (3.1.4), being identical to the r-matrix structure of the original model, (2.1.8), we introduce boundary conditions in an identical manner. That is, we start by choosing a pair of matrices, $K_\pm$, that satisfy (2.3.1). Specifically, we use the same K-matrices as in the original picture, (2.3.2):

$$K_\pm^{\text{(NLS)}}(\lambda) = K_\pm^{\text{(HM)}}(\lambda) = \alpha_\pm \mathbb{I} + \lambda \begin{pmatrix} \delta_\pm & \beta_\pm \\ \gamma_\pm & -\delta_\pm \end{pmatrix}, \qquad (3.4.1)$$

where, just as in the standard picture of Chapter 2, we focus only on non-dynamical boundary conditions. We introduce these K-matrices into the generator of the spatially conserved quantities as [20, 11]:

$$\bar{\mathfrak{t}}_T(\lambda) = \text{tr}\left\{ K_+(\lambda) T_T(\tau, -\tau; \lambda) K_-(\lambda) T_T^{-1}(\tau, -\tau; -\lambda) \right\}, \qquad (3.4.2)$$

from which we can use the quadratic relation (3.2.2) and the defining relation for the K-matrices, (2.3.1), to derive the time-like equivalent of (2.3.3), which tells us that the $\bar{\mathfrak{t}}_T$ Poisson commute for different spectral parameters. Again, we are actually interested in the coefficients in the expansion of $\bar{\mathcal{G}}_T(\lambda) = \ln\left(\bar{\mathfrak{t}}_T(\lambda) \right)$, which will also Poisson commute with one another:

$$\{\bar{\mathcal{G}}_T^{(k)}, \bar{\mathcal{G}}_T^{(j)}\}_T = 0. \qquad (3.4.3)$$

In order to evaluate the series expansion of $\bar{\mathcal{G}}_T(\lambda)$, as well as diagonalising T_T through (3.2.1), we need to also diagonalise T_T^{-1} through:

$$T_T^{-1}(t_1, t_2; -\lambda) = \left(\mathbb{I} + W_T(t_2; -\lambda) \right) e^{-Z_T(t_1, t_2; -\lambda)} \left(\mathbb{I} + W_T(t_1; -\lambda) \right)^{-1}. \qquad (3.4.4)$$

An important point here is that when we take the appropriate limit in λ of the exponentiated term, due to the $-$ sign in front of the Z_T and the highest order term

being $(-\lambda)^{\pm 2} = \lambda^{\pm 2}$, the dominant term in the limit as λ goes to ∞ (for NLS) or 0^+ (for HM) will instead (contrasted against (2.3.6)) be:

$$\mathrm{e}^{-Z_T(t_1,t_2;-\lambda)} \to \mathrm{e}^{-Z_{22,T}(t_1,t_2;-\lambda)}e_{22} + \mathcal{O}(\mathrm{e}^{-\lambda^{\mp 2}}). \tag{3.4.5}$$

Consequently, when the diagonalisations are inserted into the generator $\bar{\mathcal{G}}_T$, we have (where we suppress the parameters by defining $\hat{\mathfrak{f}} = \mathfrak{f}(-\lambda)$ and $W_{\pm,T} = W_T(\pm\tau)$):

$$\bar{\mathcal{G}}_T(\lambda) = \ln\left(\mathrm{e}^{Z_{11,T}-\hat{Z}_{22,T}}\mathrm{tr}\left\{K_+\left(\mathbb{I}+W_{+,T}\right)e_{11}\left(\mathbb{I}+W_{-,T}\right)^{-1}\right.\right.$$
$$\left.\left.\times K_-\left(\mathbb{I}+\hat{W}_{-,T}\right)e_{22}\left(\mathbb{I}+\hat{W}_{+,T}\right)^{-1}\right\}\right), \tag{3.4.6}$$

which can be separated into the bulk contribution and the two boundary contributions:

$$\bar{\mathcal{G}}_T(\lambda) = Z_{11,T}(\lambda) - Z_{22,T}(-\lambda) + \ln\left(\mathbb{W}_+(\lambda)\right) + \ln\left(\mathbb{W}_-(\lambda)\right), \tag{3.4.7}$$

where we define:

$$\mathbb{W}_+(\lambda) = \left[\left(\mathbb{I}+W_T(\tau;-\lambda)\right)^{-1}K_+(\lambda)\left(\mathbb{I}+W_T(\tau;\lambda)\right)\right]_{21},$$
$$\mathbb{W}_-(\lambda) = \left[\left(\mathbb{I}+W_T(-\tau;\lambda)\right)^{-1}K_-(\lambda)\left(\mathbb{I}+W_T(-\tau;-\lambda)\right)\right]_{12}. \tag{3.4.8}$$

Non-Linear Schrödinger

The bulk contributions to the open integrals of motion are found by using the Z_T-matrices from (3.2.8) in (3.4.7):

$$\bar{\mathcal{G}}_T^{(\mathrm{NLS},-2)} = 2\tau, \qquad\qquad \bar{\mathcal{G}}_T^{(\mathrm{NLS},-1)} = \bar{\mathcal{G}}_T^{(\mathrm{NLS},0)} = \bar{\mathcal{G}}_T^{(\mathrm{NLS},1)} = 0,$$

$$\bar{\mathcal{G}}_T^{(\mathrm{NLS},2)} = \int_{-\tau}^{\tau}\left[2\left(|\psi|^4 - |\phi|^2\right) - \left(\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}\right)\right]\mathrm{d}t, \tag{3.4.9}$$

$$\bar{\mathcal{G}}_T^{(\mathrm{NLS},3)} = \int_{-\tau}^{\tau}\left(\left(\bar{\psi}\dot{\phi} + \dot{\bar{\psi}}\phi\right) - \left(\dot{\psi}\bar{\phi} + \psi\dot{\bar{\phi}}\right)\right)\mathrm{d}t.$$

Due to the logarithmic dependence of $\bar{\mathcal{G}}_T$ on $\mathbb{W}_\pm$, the lowest order contribution to

the conserved quantities will appear at order λ^0. As the conserved quantity at this order is trivial, we expect this to be constant. Indeed, the first three contributions from the boundary terms are:

$$\ln\left(\mathbb{W}_+\right)^{(\text{NLS},0)} = \ln\left(\gamma_+\right), \qquad \ln\left(\mathbb{W}_+\right)^{(\text{NLS},1)} = 0,$$

$$\ln\left(\mathbb{W}_+\right)^{(\text{NLS},2)} = \frac{1}{\gamma_+}\left(2\alpha_+\psi + 2\delta_+\phi + \psi(\beta_+\psi - \gamma_+\bar{\psi})\right), \qquad (3.4.10)$$

$$\ln\left(\mathbb{W}_+\right)^{(\text{NLS},3)} = \psi\bar{\phi} - \phi\bar{\psi},$$

and:

$$\ln\left(\mathbb{W}_-\right)^{(\text{NLS},0)} = \ln\left(\beta_-\right), \qquad \ln\left(\mathbb{W}_-\right)^{(\text{NLS},1)} = 0,$$

$$\ln\left(\mathbb{W}_-\right)^{(\text{NLS},2)} = \frac{1}{\beta_-}\left(2\alpha_-\bar{\psi} - 2\delta_-\bar{\phi} - \bar{\psi}(\beta_-\psi - \gamma_-\bar{\psi})\right), \qquad (3.4.11)$$

$$\ln\left(\mathbb{W}_-\right)^{(\text{NLS},3)} = \phi\bar{\psi} - \psi\bar{\phi}.$$

By combining these boundary contributions with the bulk conserved quantities above, we can extract the dual open NLS Hamiltonian from the order λ^{-2} terms:

$$\bar{H}_T^{(\text{NLS})} = \int_{-\tau}^{\tau}\left[2\left(|\psi|^4 - |\phi|^2\right) - \left(\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}\right)\right]\mathrm{d}t$$

$$+ \frac{1}{\gamma_+}\left(2\alpha_+\psi + 2\delta_+\phi + \psi(\beta_+\psi - \gamma_+\bar{\psi})\right) \qquad (3.4.12)$$

$$+ \frac{1}{\beta_-}\left(2\alpha_-\bar{\psi} - 2\delta_-\bar{\phi} - \bar{\psi}(\beta_-\psi - \gamma_-\bar{\psi})\right).$$

Away from the boundaries, the Poisson brackets of $\bar{H}_T^{(\text{NLS})}$ with each of the four fields returns the space-evolution equations, (3.2.14). At the boundaries, however, when the space-evolution is derived the condition that the fields at the boundary still satisfy the usual space-evolution equations imposes extra conditions on the fields ψ, $\bar{\psi}$, ϕ, and $\bar{\phi}$, as well as the boundary fields $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, and $\delta_\pm$. These conditions are the time-like boundary conditions for the NLS model:

$$\delta_\pm = 0, \qquad \gamma_+ = \beta_- = 1,$$

$$\bar{\psi}(+\tau) = \alpha_+ + \beta_+\psi(+\tau), \qquad \psi(-\tau) = \alpha_- + \gamma_-\bar{\psi}(-\tau). \qquad (3.4.13)$$

Isotropic Landau-Lifshitz

The dual construction of the HM model is comparatively straight-forward, as the dual Hamiltonian appears at order λ^0 in the hierarchy, so following the reasoning from our NLS considerations we only need to find the first order contribution from the boundary terms, $\ln\left(\mathbb{W}_\pm\right)$. The calculation is kept non-trivial, however, by the fact that the order λ^0 term in the expansion of $\mathbb{W}_\pm$ is 0, so that the actual lowest order term is:

$$\mathbb{W}_\pm^{(\text{HM},1)} = \frac{1}{2c}\left(\frac{\pm 2\alpha_\pm}{c}\left(\frac{S_\pm\Sigma_z}{S_z + c} - \Sigma_\pm\right) - 2\delta_\pm S_\pm - \beta_\pm \frac{S_+ S_\pm}{S_z \pm c} - \gamma_\pm \frac{S_- S_\pm}{S_z \mp c}\right), \quad (3.4.14)$$

so that the first three terms in the expansion of $\bar{\mathcal{G}}_T$ are:

$$\bar{\mathcal{G}}_T^{(\text{HM},-2)} = 2c\tau, \qquad\qquad \bar{\mathcal{G}}_T^{(\text{HM},-1)} = 0, \qquad\qquad (3.4.15)$$

$$\bar{\mathcal{G}}_T^{(\text{HM},0)} = \frac{1}{2c}\int_{-\tau}^{\tau}\left(\frac{S_+\dot{S}_- - \dot{S}_+ S_-}{c + S_z} - \frac{1}{c^2}\left(\Sigma_+\Sigma_- + \Sigma_z^2\right)\right)\mathrm{d}t + \ln\left(\mathbb{W}_+^{(\text{HM},1)}\mathbb{W}_-^{(\text{HM},1)}\right).$$

Multiplying $\bar{\mathcal{G}}_T^{(\text{HM},0)}$ by the factor $-c$ gives the Hamiltonian with open boundary conditions:

$$\bar{H}_T^{(\text{HM})} = \int_{-\tau}^{\tau}\left(\frac{1}{2c^2}\left(\Sigma_+\Sigma_- + \Sigma_z^2\right) + \frac{\dot{S}_+ S_- - S_+ \dot{S}_-}{2(c + S_z)}\right)\mathrm{d}t - c\ln\left(\mathbb{W}_+^{(\text{HM},1)}\mathbb{W}_-^{(\text{HM},1)}\right).$$
$$(3.4.16)$$

Using the same approach as for NLS, the boundary conditions can be extracted from combining this Hamiltonian with the dual Poisson brackets. The requirement that $\lim_{t\to\pm\tau} S'_\sigma = S'_\sigma(\pm\tau)$ restricts us to the case $\alpha_\pm = 0$. If we combine this with the requirement that $\lim_{t\to\pm\tau}\Sigma'_\sigma = \Sigma'_\sigma(\pm\tau)$, then we find the time-like boundary conditions for the HM model:

$$\alpha_\pm = 0, \qquad\qquad \beta_\pm S_+ + \gamma_\pm S_- + 2\delta_\pm S_z = 0. \qquad\qquad (3.4.17)$$

3.4.2 *U*-Matrices

We can also find a generator for the U-matrices both in the bulk and at the boundaries. The generator for the bulk U-matrices will be (where we suppress the param-

eter dependence on the left-hand side for compactness) [11]:

$$\bar{\mathbb{U}}_{\mathrm{B},b} = \bar{\mathfrak{t}}_T^{-1}(\mu)\mathrm{tr}_a\left\{K_{+,a}(\mu)T_{T,a}(\tau,t;\mu)r_{ab}(\mu-\lambda)T_{T,a}(t,-\tau;\mu)K_{-,a}(\mu)T_{T,a}^{-1}(-\mu)\right.$$

$$\left. + K_{+,a}(\mu)T_{T,a}(\mu)K_{-,a}(\mu)T_{T,a}^{-1}(t,-\tau;-\mu)r_{ab}(\mu+\lambda)T_{T,a}^{-1}(\tau,t;-\mu)\right\},$$

$$(3.4.18)$$

and, being mindful of the different limit for the $T_T^{-1}(-\mu)$, this can be reduced to:

$$\bar{\mathbb{U}}_{\mathrm{B}}^{(\mathrm{NLS})}(t;\lambda,\mu) = \mathbb{U}^{(\mathrm{NLS})}(t;\lambda,\mu) + \frac{1}{\mu+\lambda}\big(\mathbb{I}+W_T(t;-\mu)\big)e_{22}\big(\mathbb{I}+W_T(t;-\mu)\big)^{-1},$$

$$(3.4.19)$$

where $\mathbb{U}(t;\lambda,\mu)$ is the generator of the U-matrices with periodic boundary conditions. The generator for the HM model is the same, except with $\frac{1}{\mu+\lambda}\rightarrow\frac{1}{2(\mu+\lambda)}$. Unlike in the original case, where the second term differed from the first only by the sign of the μ, here it differs both by the sign of the μ and in that the matrix e_{11} has become e_{22}. The boundary U-matrices are found by considering the generators:

$$\bar{\mathbb{U}}_{+,b}(\lambda,\mu) = \bar{\mathfrak{t}}_T^{-1}(\mu)\mathrm{tr}_a\left\{K_{-,a}(\mu)T_{T,a}^{-1}(-\mu)K_{+,a}(\mu)r_{ab}(\mu-\lambda)T_{T,a}(\mu)\right.$$

$$\left.+K_{-,a}(\mu)T_{T,a}^{-1}(-\mu)r_{ab}(\mu+\lambda)K_{+,a}(\mu)T_{T,a}(\mu)\right\},$$

$$(3.4.20)$$

$$\bar{\mathbb{U}}_{-,b}(\lambda,\mu) = \bar{\mathfrak{t}}_T^{-1}(\mu)\mathrm{tr}_a\left\{K_{+,a}(\mu)T_{T,a}(\mu)r_{ab}(\mu-\lambda)K_{-,a}(\mu)T_{T,a}^{-1}(-\mu)\right.$$

$$\left.+K_{+,a}(\mu)T_{T,a}(\mu)K_{-,a}(\mu)r_{ab}(\mu+\lambda)T_{T,a}^{-1}(-\mu)\right\},$$

which, for NLS, can be simplified to:

$$\bar{\mathbb{U}}_{+,b}^{(\mathrm{NLS})}(\lambda,\mu) = \frac{1}{\mathbb{W}_+(\mu)}\left(\frac{1}{\mu-\lambda}\big(\mathbb{I}+W_T(\tau;\mu)\big)e_{12}\big(\mathbb{I}+W_T(\tau;-\mu)\big)^{-1}K_+(\mu)\right.$$

$$\left.+\frac{1}{\mu+\lambda}K_+(\mu)\big(\mathbb{I}+W_T(\tau;\mu)\big)e_{12}\big(\mathbb{I}+W_T(\tau;-\mu)\big)^{-1}\right),$$

$$(3.4.21)$$

and:

$$\bar{\mathbb{U}}_{-,b}^{(\mathrm{NLS})}(\lambda,\mu) = \frac{1}{\mathbb{W}_-(\mu)}\left(\frac{1}{\mu-\lambda}K_-(\mu)\big(\mathbb{I}+W_T(-\tau;-\mu)\big)e_{21}\big(\mathbb{I}+W_T(-\tau;\mu)\big)\right)^{-1}$$

$$+\frac{1}{\mu+\lambda}\big(\mathbb{I}+W_T(-\tau;-\mu)\big)e_{21}\big(\mathbb{I}+W_T(-\tau;\mu)\big)^{-1}K_-(\mu)\bigg).$$

$$(3.4.22)$$

The boundary generators for the HM model are relatedly $\bar{\mathbb{U}}_{\pm,b}^{(HM)} = \frac{1}{2}\bar{\mathbb{U}}_{\pm,b}^{(NLS)}$.

Non-Linear Schrödinger

Just as the first non-trivial conserved quantity in (3.4.9) is the Hamiltonian at order λ^{-2}, the first non-trivial U-matrix is the one corresponding to the Hamiltonian. Also, as the order λ^{-3} conserved quantity is a total derivative, this turns out to also be trivial. Consequently, the only open U-matrices we report here are the order λ^{-2} ones. In the bulk, these are:

$$\mathbb{U}_{B}^{(NLS,2)} = \begin{pmatrix} \lambda & 2\bar{\psi} \\ 2\psi & -\lambda \end{pmatrix}, \tag{3.4.23}$$

while at the $t = +\tau$ boundary they are:

$$\mathbb{U}_{+}^{(NLS,2)} = \frac{1}{\gamma_+} \begin{pmatrix} \lambda\gamma_+ + 2\delta_+\psi & 2\alpha_+ - 2\delta_+\lambda + 2\beta_+\psi \\ 2\psi & -\lambda\gamma_+ - 2\delta_+\psi \end{pmatrix}, \tag{3.4.24}$$

and at the $t = -\tau$ boundary:

$$\mathbb{U}_{-}^{(NLS,2)} = \frac{1}{\beta_-} \begin{pmatrix} \lambda\beta_- + 2\delta_-\bar{\psi} & 2\bar{\psi} \\ 2\alpha_- - 2\lambda\delta_- + 2\gamma_-\bar{\psi} & -\lambda\beta_- - 2\delta_-\bar{\psi} \end{pmatrix}. \tag{3.4.25}$$

Requiring that $\lim_{t\to\pm\tau} \mathbb{U}_{B}^{(NLS,2)} = \mathbb{U}_{\pm}^{(NLS,2)}$ gives rise to the same boundary conditions as those found from the Hamiltonian approach, (3.4.13).

Isotropic Landau-Lifshitz

The lowest order term in the expansion of the bulk generator appears as the coefficient of μ^0, and is:

$$\mathbb{U}_{B}^{(HM,0)} = \frac{-1}{2c\lambda} \begin{pmatrix} S_z & S_- \\ S_+ & -S_z \end{pmatrix} = 2\mathbb{U}^{(HM,0)}, \tag{3.4.26}$$

where $\mathbb{U}^{(HM,0)}$ is the U-matrix appearing at lowest order in the periodic case. The first non-trivial terms in the expansion of each of the boundary generators also

appear at order μ^0. For the $t = +\tau$ boundary, this is:

$$\mathbb{U}_+^{(\mathrm{HM},0)} = \frac{1}{2c(c+S_z)\mathbb{W}_+^{(1)}} \left[\frac{\alpha_+}{\lambda^2} \begin{pmatrix} S_+(c+S_z) & -(c+S_z)^2 \\ S_+^2 & -S_+(c+S_z) \end{pmatrix} \right.$$

$$\left. - \frac{1}{2\lambda} \begin{pmatrix} -\beta_+ S_+^2 - \gamma_+(c+S_z)^2 & 2(c+S_z)\big(\delta_+(c+S_z) + \beta_+ S_+\big) \\ 2S_+\big(\delta_+ S_+ - \gamma_+(c+S_z)\big) & \beta_+ S_+^2 + \gamma_+(c+S_z)^2 \end{pmatrix} \right],$$

$$(3.4.27)$$

while at the $t = -\tau$ boundary, it is:

$$\mathbb{U}_-^{(\mathrm{HM},0)} = \frac{1}{2c(c+S_z)\mathbb{W}_-^{(1)}} \left[\frac{\alpha_-}{\lambda^2} \begin{pmatrix} S_-(c+S_z) & S_-^2 \\ -(c+S_z)^2 & -S_-(c+S_z) \end{pmatrix} \right.$$

$$\left. - \frac{1}{2\lambda} \begin{pmatrix} \beta_-(c+S_z)^2 + \gamma_- S_-^2 & -2S_-\big(\delta_- S_- - \beta_-(c+S_z)\big) \\ -2(c+S_z)\big(\delta_-(c+S_z) + \gamma_- S_-\big) & -\beta_-(c+S_z)^2 - \gamma_- S_-^2 \end{pmatrix} \right].$$

$$(3.4.28)$$

As with NLS, requiring that $\lim_{t\to\pm\tau} \mathbb{U}_\mathrm{B}^{(\mathrm{HM},0)} = \mathbb{U}_\pm^{(\mathrm{HM},0)}$ gives rise to both the condition that $\alpha_\pm = 0$ (from the order λ^{-2} terms) and that $\beta_\pm S_+ + \gamma_\pm S_- + 2\delta_\pm S_z = 0$, which agrees with the boundary conditions found from the Hamiltonian approach, (3.4.17).

By comparing the time-like boundary conditions, (3.4.13) and (3.4.17), with the respective associated space-like boundary conditions, (2.3.17) and (2.3.18), we can see that there is no evident connection between the two. This asymmetry is rooted in the fundamentally different dependence of the fields on the space and time coordinates, as can be seen by comparing the forms of the equations of motion in (1.1.1) against (3.2.14), and (1.1.3) against (3.2.17).

Chapter 4

The Quantum Auxiliary Linear Problem

The theory of quantum spin chains is a well studied topic, with many of the tools used in the previous chapters (such as the r-matrix and boundary K-matrices) having originated in the quantum setting, and been translated into a classical context through suitable limiting procedures. However, while the L-matrix (we only discuss discrete quantum systems, i.e. spin chains here, so the discrete classical systems of Section 2.4 are the closest classical parallel) alone does appear in the techniques of quantum integrable spin chains, such as the quantum inverse scattering method or the Bethe ansatz, its temporal partner (the A-matrix) is not commonly discussed. This was in part due to the lack of a systematic analogue of the classical Semenov-Tian-Shansky (STS) formula, (2.5.13). Such a formula was derived for both closed, (4.1.16), and open, (4.3.15), spin chains in [14] (although the case of closed boundary conditions was previously discussed in [33], and a similar result to that from [14] was found) and is primarily what we address in this chapter.

The first thing we do therefore is derive a suitable quantum STS formula in Section 4.1, then give examples in Section 4.2 with the XXX Heisenberg spin chain (the model which the isotropic Landau-Lifshitz model studied in previous chapters is the continuous classical analogue of) and the quantum Ablowitz-Ladik (qAL) model [26], which is to be contrasted against the Ablowitz-Ladik model described in Section 2.4.

As with the previous chapters, we then repeat the process for spin chains with

open boundary conditions in Section 4.3 and work through the results for the open XXX and qAL models (the qAL model has previously been studied in the context of open boundary conditions in [34]) in Section 4.4. It is in these open models that we see particular non-trivial quantum corrections arising.

This chapter is built out of the paper [14], with the addition of examples using the XXX model.

4.1   Time Evolution in Closed Systems

The quantum L-matrix obeys the spatial half of the semi-discrete auxiliary-linear problem, (2.4.2), with some auxiliary vector Ψ_{an}:

$$\Psi_{a,n+1} = L_{an}\Psi_{an}, \tag{4.1.1}$$

so when attempting to introduce an A-matrix that describes the time-evolution of the system, we do so in an equivalent manner to the temporal half of the auxiliary linear problem:

$$\partial_t \Psi_{an} = A_{an}\Psi_{an}. \tag{4.1.2}$$

The goal of this section is then to find a closed form expression for a generator, $\mathbb{A}_n$, of the A-matrices associated to a model described by a given L-matrix and r-matrix.

The Yang-Baxter and RLL Equations

The fundamental structure underlying quantum integrable models comes from the Yang-Baxter equation [4]:

$$R_{ab}(\lambda - \mu)R_{ac}(\lambda)R_{bc}(\mu) = R_{bc}(\mu)R_{ac}(\lambda)R_{ab}(\lambda - \mu), \tag{4.1.3}$$

where R is a matrix that acts on two copies of an underlying vector space, $V \otimes V$, with the subscripts denoting which two copies it acts on, so that the whole equation acts on $V \otimes V \otimes V$. The R-matrices are allowed to depend on some additional free parameter, denoted by either λ or μ, called the spectral parameter.

In the classical limit $R \to \mathbb{I} + i\hbar r + ...$, the Yang-Baxter equation becomes the classical Yang-Baxter equation, (2.1.6), which was used for the classical models studied in previous chapters.

In the quantum setting (in the Heisenberg picture), the time evolution of the operators in a system is given by Heisenberg's equation:

$$\partial_t \mathcal{O} = i[H, \mathcal{O}], \tag{4.1.4}$$

where $\mathcal{O}$ is the operator in question and H is the Hamiltonian. Thus, if we want to find the time evolution of a system, we need to first find expressions for the commutators between the fields of the model. We do this through the RLL relation:

$$R_{ab}(\lambda - \mu)L_{an}(\lambda)L_{bn}(\mu) = L_{bn}(\mu)L_{an}(\lambda)R_{ab}(\lambda - \mu), \tag{4.1.5}$$

which can be recognised as the Yang-Baxter equation, (4.1.3), but with the c vector space distinguished (and labelled here as n) so that the matrices acting on it become Lax matrices. It will be beneficial to rewrite this in a more explicit manner:

$$\left[L_{an}(\lambda), L_{bm}(\mu)\right] = \left(\mathfrak{R}_{ab}(\lambda-\mu)L_{an}(\lambda)L_{bn}(\mu) - L_{bn}(\mu)L_{an}(\lambda)\mathfrak{R}_{ab}(\lambda-\mu)\right)\delta_{nm}, \tag{4.1.6}$$

where we define for compactness $\mathfrak{R}_{ab}(\lambda) = \mathbb{I}_{ab} - R_{ab}(\lambda)$. The Kronecker delta factor is introduced to more generally describe the commutativity of matrices acting on wholly different spaces.

Generating Commuting Quantities

The generation of commuting quantities now follows the same process as for classical discrete models, where we first define the transport matrix:

$$T_a(n, m; \lambda) = L_{an}(\lambda)...L_{am}(\lambda), \tag{4.1.7}$$

with $n > m$, and then the monodromy matrix as $T_a = T_a(N, 1)$. The monodromy matrix satisfies an RTT relation, as can be found by using how objects acting on

wholly different spaces commute with one another (so that $L_{an}L_{bm} = L_{bm}L_{an}$ for $a \neq b$ and $n \neq m$) [18, 19]:

$$R_{ab}(\lambda - \mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda - \mu). \tag{4.1.8}$$

The transfer matrix $\mathfrak{t}$ is then defined as the trace of the monodromy matrix, $\mathfrak{t}(\lambda) = \mathrm{tr}_a\{T_a(\lambda)\}$, and by making use of the RTT relation we can see that this commutes with itself for different values of the spectral parameter. Consequently, if this is expanded as a power series in the spectral parameter, the coefficients $\mathfrak{t}^{(k)}$ of λ^k will commute with one another:

$$\left[\mathfrak{t}^{(k)}, \mathfrak{t}^{(j)}\right] = 0. \tag{4.1.9}$$

As these coefficients all commute with one another, the logarithm of the transfer matrix can be considered, $\mathcal{G}(\lambda) = \ln(\mathfrak{t}(\lambda))$, and the coefficients in the series expansion of that commute with one another as well:

$$\left[\mathcal{G}^{(k)}, \mathcal{G}^{(j)}\right] = 0. \tag{4.1.10}$$

Each of the quantities generated in either of these ways can be treated as the Hamiltonian governing the evolution along a distinct time flow. Then, as we know that they commute with one another, this tells us that the other quantities will all be constant in this system.

4.1.1   The A-Matrix Generator

In parallel to how the classical STS formula is derived, we start by using the transfer matrix in place of the Hamiltonian:

$$\partial_{\bar{t}} L_{bn}(\mu) = \mathrm{i}\left[\mathfrak{t}(\lambda), L_{bn}(\mu)\right], \tag{4.1.11}$$

where $\bar{t}$ denotes the "universal time" that contains all of the distinct time flows, $\bar{t} = \sum_n \lambda^n t_n$. Then, as the L_{an} commute with L_{bm} when both $a \neq b$ and $n \neq m$ (that

is, they act ultra-locally), the only term in the $\mathfrak{t}$ that interacts with the commutator will be L_{an}:

$$\partial_{\bar{t}} L_{bn}(\mu) = \mathrm{i} \mathrm{tr}_a \left\{ T_a(N, n+1; \lambda) \left[L_{an}(\lambda), L_{bn}(\mu) \right] T_a(n-1, 1; \lambda) \right\}. \qquad (4.1.12)$$

Using the alternate form of the RLL relation, (4.1.6), we can evaluate this commutator:

$$\partial_{\bar{t}} L_{bn}(\mu) = \mathrm{i} \mathrm{tr}_a \left\{ T_a(N, n+1; \lambda) \mathfrak{R}_{ab}(\lambda - \mu) T_a(n, 1; \lambda) \right\} L_{bn}(\mu)$$

$$\qquad (4.1.13)$$

$$- \mathrm{i} L_{bn}(\mu) \mathrm{tr}_a \left\{ T_a(N, n; \lambda) \mathfrak{R}_{ab}(\lambda - \mu) T_a(n-1, 1; \lambda) \right\},$$

however, by comparing this with the compatibility condition of the two halves of the quantum auxiliary linear problem we are constructing, (4.1.1) and (4.1.2):

$$\partial_t L_{an} = A_{a,n+1} L_{an} - L_{an} A_{an}, \qquad (4.1.14)$$

it is evident that the generator of the A-matrices is written as [14]:

$$\mathbb{A}_{bn}(\lambda, \mu) = \mathrm{i} \mathrm{tr}_a \left\{ T_a(N, n; \lambda) \mathfrak{R}_{ab}(\lambda - \mu) T_a(n-1, 1; \lambda) \right\}, \qquad (4.1.15)$$

or by recalling the definition of $\mathfrak{R}$:

$$\mathbb{A}_{bn}(\lambda, \mu) = \mathrm{i} \mathfrak{t}(\lambda) \mathbb{I} - \mathrm{i} \mathbb{B}_{bn}(\lambda, \mu), \qquad (4.1.16)$$

where we define:

$$\mathbb{B}_{bn}(\lambda, \mu) = \mathrm{tr}_a \left\{ T_a(N, n; \lambda) R_{ab}(\lambda - \mu) T_a(n-1, 1; \lambda) \right\}. \qquad (4.1.17)$$

To find the A-matrix associated to the Hamiltonian $\mathfrak{t}^{(k)}$, we simply expand (4.1.17) about powers of λ and take the coefficient of order λ^k, labelled $\mathbb{B}_{bn}^{(k)}$, and combine it with the Hamiltonian as:

$$\mathbb{A}_{bn}^{(k)}(\mu) = \mathrm{i} \mathfrak{t}^{(k)} \mathbb{I} - \mathrm{i} \mathbb{B}_{bn}^{(k)}(\mu). \qquad (4.1.18)$$

If we now insert this into the zero-curvature condition, (4.1.14), then the terms explicitly containing $\mathfrak{t}^{(k)}$ are simply the right-hand side of Heisenberg's equation. Consequently, it follows that the $\mathbb{B}_{bn}^{(k)}$ are only shifted by the action of the L-matrix and leave L_{an} unchanged, i.e. $L_{an}\mathbb{B}_{an}^{(k)} = \mathbb{B}_{a,n+1}^{(k)}L_{an}$. This can in fact be shown to hold for the entire generator $\mathbb{B}_{bn}$ by using the RLL relation, (4.1.5):

$$L_{an}(\mu)\mathbb{B}_{an}(\lambda, \mu) = \mathbb{B}_{a,n+1}(\lambda, \mu)L_{an}(\mu). \tag{4.1.19}$$

Logarithmic Generator

As mentioned when defining the generator of the commuting quantities, we are actually interested in the quantities generated by $\mathcal{G}(\lambda)$. Unfortunately, however, the process for finding the A-matrix generator in this quantum setting is not as simple as it was for the equivalent classical case, since we can no longer use the fact that $[\ln(a), b] = a^{-1}[a, b]$ due to the non-commutativity of $\mathfrak{t}$ and $[\mathfrak{t}, \mathcal{O}]$. Instead, we choose to introduce the B-matrix dependence of the A-matrices with the intent of having the classical limit agree. Specifically, we choose the generator of the A-matrices corresponding to the Hamiltonians generated by $\mathcal{G}$ to be:

$$\mathbb{A}_{bn}(\lambda, \mu) = \mathrm{i}\mathcal{G}(\lambda)\mathbb{I} - \mathrm{i}\ln\left(\mathbb{B}_{bn}(\lambda, \mu)\right). \tag{4.1.20}$$

Care needs to be taken here however, as the potential non-commutativity of the $\mathbb{B}_{bn}$ inside the logarithm could cause this to not be well defined. Fortunately, we can show that the different terms in the expansion of $\mathbb{B}$ do actually commute with one another so that this is not a problem. We start by considering the product of two such B-matrix generators and commute the terms that do not affect each other such that the result is in a convenient form (where we also define $T^+ = T(N, n)$ and $T^- = T(n-1, 1)$):

$$\mathbb{B}_{cn}(\mu, \lambda)\mathbb{B}_{cn}(\xi, \lambda) = \mathrm{tr}_{ab}\left\{T_a^+(\mu)R_{ac}(\mu - \lambda)T_a^-(\mu)T_b^+(\xi)R_{bc}(\xi - \lambda)T_b^-(\xi)\right\}$$

$$= \mathrm{tr}_{ab}\left\{T_a^+(\mu)T_b^+(\xi)R_{ac}(\mu - \lambda)R_{bc}(\xi - \lambda)T_a^-(\mu)T_b^-(\xi)\right\}. \tag{4.1.21}$$

Then, if we use the Yang-Baxter equation, (4.1.3), to reverse the order of the two

R-matrices in this expression, we have:

$$= \text{tr}_{ab} \left\{ T_a^+(\mu) T_b^+(\xi) \left(R_{ab}(\mu - \xi) \right)^{-1} R_{bc}(\xi - \lambda) R_{ac}(\mu - \lambda) R_{ab}(\mu - \xi) T_a^-(\mu) T_b^-(\xi) \right\}.$$
(4.1.22)

However, we can also use the RTT relations satisfied by the $T^\pm$, (4.1.8), to swap the order of the $T_a^\pm$ and $T_b^\pm$ with the help of the extra R-matrices from the Yang-Baxter equation:

$$= \text{tr}_{ab} \left\{ \left(R_{ab}(\mu - \xi) \right)^{-1} T_b^+(\xi) T_a^+(\mu) R_{bc}(\xi - \lambda) R_{ac}(\mu - \lambda) T_b^-(\xi) T_a^-(\mu) R_{ab}(\mu - \xi) \right\}.$$
(4.1.23)

Finally, as this is all inside a partial trace over the a and b spaces, we can cycle the R_{ab} so that it cancels out with its inverse. Rearranging the terms in the middle that commute with each other we can return this expression to a familiar form:

$$= \text{tr}_{ab} \left\{ T_b^+(\xi) R_{bc}(\xi - \lambda) T_b^-(\xi) T_a^+(\mu) R_{ac}(\mu - \lambda) T_a^-(\mu) \right\}$$
(4.1.24)
$$= \mathbb{B}_{cn}(\xi, \lambda) \mathbb{B}_{cn}(\mu, \lambda),$$

which proves that $\left[\mathbb{B}_{an}^{(k)}, \mathbb{B}_{an}^{(j)} \right] = 0$ for all k, j. Thus, it makes sense to talk about the logarithm of these B-matrices, as we can factor the lowest order term (assumed to be order k) out:

$$\ln \left(\mathbb{B}_{bn}(\lambda, \mu) \right) = \ln \left(\mathbb{B}_{bn}^{(k)}(\mu) \right) + \ln \left(\mathbb{I} + \left(\mathbb{B}_{bn}^{(k)}(\mu) \right)^{-1} \mathcal{O}(\lambda) \right),$$
(4.1.25)

whereupon the logarithm can be expanded for $\lambda \to 0$. A similar argument holds for other choices of λ limit.

The Classical Limit

The classical limit we consider is where $R_{ab}(\lambda) \to \mathbb{I} + i\hbar r_{ab}(\lambda) + ...$ and $L_{an} \to \hbar L_{an}$. As mentioned earlier, this limit converts the Yang-Baxter equation, (4.1.3), to its classical counterpart, (2.1.6). It also converts the algebraic relation (4.1.6) to the quadratic Poisson bracket relation (2.4.5) when we identify $(i\hbar)^{-1}[\cdot, \cdot] \to \{\cdot, \cdot\}$.

The action of this limit on the generator of the B-matrices is as follows:

$$\mathbb{B}_{bn}(\lambda, \mu) \to \hbar^N \mathrm{tr}_a \left\{ T_a(N, n; \lambda) \big(\mathbb{I} + i\hbar r_{ab}(\lambda - \mu) \big) T_a(n - 1, 1; \lambda) \right\} + \ldots$$

$$\to \hbar^N \mathfrak{t}(\lambda) \mathbb{I} + i\hbar^{N+1} \mathrm{tr}_a \left\{ T_a(N, n; \lambda) r_{ab}(\lambda - \mu) T_a(n - 1, 1; \lambda) \right\} + \ldots, \tag{4.1.26}$$

so that the non-logarithmic A-matrix generator, (4.1.16), becomes:

$$\mathbb{A}_{bn}(\lambda, \mu) \to \hbar^{N+1} \mathrm{tr}_a \left\{ T_a(N, n; \lambda) r_{ab}(\lambda - \mu) T_a(n - 1, 1; \lambda) \right\} + \ldots, \tag{4.1.27}$$

and the logarithmic generator (4.1.20) becomes:

$$\mathbb{A}_{bn} \to i\ln\left(\hbar^N \mathfrak{t} \right) \mathbb{I} - i\ln\left(\hbar^N \mathfrak{t} \mathbb{I} + i\hbar^{N+1} \mathrm{tr}_a \left\{ T_a(N, n) r_{ab}(\lambda - \mu) T_a(n - 1, 1) \right\} + \ldots \right)$$

$$\to -i\ln\left(\mathbb{I} + i\hbar \mathfrak{t}^{-1} \mathrm{tr}_a \left\{ T_a(N, n) r_{ab}(\lambda - \mu) T_a(n - 1, 1) \right\} + \ldots \right) \tag{4.1.28}$$

$$\to \hbar \mathfrak{t}^{-1}(\lambda) \mathrm{tr}_a \left\{ T_a(N, n; \lambda) r_{ab}(\lambda - \mu) T_a(n - 1, 1; \lambda) \right\} + \ldots.$$

The lowest order terms in each of these expansions exactly coincides with the generators for the classical A-matrices, (2.5.14) for the non-logarithmic version and (2.5.13) for the logarithmic version.

4.2 Examples of Closed A-Matrices

We now give examples of the A-matrices for two well studied models: The Heisenberg XXX spin chain (XXX) and a quantum Ablowitz-Ladik (qAL) model. First, we define the two models in Subsection 4.2.1 before deriving the A-matrices for each of these models in Subsection 4.2.2.

4.2.1 The Models of Interest

Heisenberg Spin Chain

The XXX Heisenberg spin chain is the prototypical integrable model, so is often used to test new methodologies in quantum integrable spin chains due to its simple

structure. The L-matrix for this model is:

$$L_n^{(\mathrm{XXX})}(\lambda) = \lambda \mathbb{I} + \mathrm{i} \begin{pmatrix} \frac{1}{2} + J_n^z & J_n^- \\ J_n^+ & \frac{1}{2} - J_n^z \end{pmatrix} = \lambda \mathbb{I} + \mathrm{i} \mathbb{J}_n, \qquad (4.2.1)$$

where the J_n^σ are the three fields (with $\sigma \in \{z, +, -\}$). As this is a model for a spin-$\frac{1}{2}$ system, we have the additional constraints that $(J_n^\pm)^2 = 0$ and $(J_n^z)^2 = \frac{1}{4}$. The R-matrix for this model is [5]:

$$R^{(\mathrm{XXX})}(\lambda) = \lambda \mathbb{I} + \mathrm{i} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (4.2.2)$$

Combining these in (4.1.6) we find $\mathfrak{su}_2$-type commutation relations between the fields J_n^σ:

$$[J_n^z, J_m^\pm] = \pm J_n^\pm \delta_{nm}, \qquad\qquad [J_n^+, J_m^-] = 2 J_n^z \delta_{nm}. \qquad (4.2.3)$$

Knowing these commutation relations and the constraints on $J_n^\pm$ and J_n^z, the anti-commutators between the fields can be extracted as $J_n^\pm J_n^z + J_n^z J_n^\pm = 0$ and $J_n^+ J_n^- + J_n^- J_n^+ = 1$. For the matrix $\mathbb{J}_n$ these manifest as the requirement that $\mathbb{J}_n^2 = \mathbb{I}$ and:

$$\mathbb{J}_n \mathbb{J}_m \mathbb{J}_n = \left(\frac{1}{2} + 2 J_m^z J_n^z + J_m^- J_n^+ + J_m^+ J_n^- \right) \mathbb{I} \equiv \mathbb{J}_{mn} \mathbb{I}, \qquad (4.2.4)$$

where we define $\mathbb{J}_{nm}$. This coefficient also obeys the properties $\mathbb{J}_{nm} = \mathbb{J}_{mn}$, $\mathbb{J}_{nm}^2 = 1$, and $\mathbb{J}_{nm} \mathbb{J}_{ns} \mathbb{J}_{nm} = \mathbb{J}_{ms}$.

The Hamiltonian for this model is:

$$H^{(\mathrm{XXX})} = \frac{1}{2} \sum_{n=1}^{N} (J_{n+1}^+ J_n^- + J_{n+1}^- J_n^+ + 2 J_{n+1}^z J_n^z) + \frac{N}{4}, \qquad (4.2.5)$$

and combining this with the commutation relations we can determine the evolution equations for this Hamiltonian:

$$\partial_t J_n^z = \mathrm{i} J_n^- (J_{n+1}^+ + J_{n-1}^+) - \mathrm{i} J_n^+ (J_{n+1}^- + J_{n-1}^-),$$

$$\partial_t J_n^\pm = \pm 2\mathrm{i} J_n^\pm (J_{n+1}^z + J_{n-1}^z) \mp 2\mathrm{i} J_n^z (J_{n+1}^\pm + J_{n-1}^\pm). \qquad (4.2.6)$$

These can be more compactly written in terms of the matrix $\mathbb{J}_n$ as:

$$\partial_t \mathbb{J}_n = \mathrm{i}\big[\mathbb{J}_n, \mathbb{J}_{n+1} + \mathbb{J}_{n-1}\big]. \tag{4.2.7}$$

To find the commuting quantities, we first expand the monodromy matrix:

$$T^{(\mathrm{XXX})}(\lambda) = \lambda^N \mathbb{I} + \mathrm{i}\lambda^{N-1}\sum_{n=1}^{N}\mathbb{J}_n + ... + \mathrm{i}^{N-1}\lambda\sum_{n=1}^{N}\mathbb{J}_N...\mathbb{J}_{n+1}\mathbb{J}_{n-1}...\mathbb{J}_1 + \mathrm{i}^N\mathbb{J}_N...\mathbb{J}_1. \tag{4.2.8}$$

In order to be able to calculate the terms at the λ^0 end of this (which are the terms we are interested in), we need to use the properties of the $\mathbb{J}_n$ noted above. Specifically, that we can write $\mathbb{J}_m\mathbb{J}_n = \mathbb{J}_n\mathbb{J}_{nm}$. Using this, we can reverse the order of the $\mathbb{J}_n$ in the lowest order term in the transfer matrix:

$$\mathfrak{t}^{(\mathrm{XXX},0)} = \mathrm{i}^N\mathrm{tr}_a\left\{\mathbb{J}_{a1}\mathbb{J}_{12}\mathbb{J}_{23}...\mathbb{J}_{N-2,N-1}\mathbb{J}_{N-1,N}\right\}. \tag{4.2.9}$$

Then, as we are tracing over the matrix space (labelled a) only the left-most $\mathbb{J}_n$ is affected, so that the two lowest order terms in the transfer matrix are:

$$\mathfrak{t}^{(\mathrm{XXX},0)} = \mathrm{i}^N\mathbb{J}_{12}\mathbb{J}_{23}...\mathbb{J}_{N-1,N-2}\mathbb{J}_{N-1,N}. \tag{4.2.10}$$

$$\mathfrak{t}^{(\mathrm{XXX},1)} = \mathrm{i}^{N-1}\left(\sum_{n=2}^{N-1}\mathbb{J}_{12}...\mathbb{J}_{n-1,n+1}...\mathbb{J}_{N-1,N} + \mathbb{J}_{23}...\mathbb{J}_{N-1,N} + \mathbb{J}_{12}...\mathbb{J}_{N-2,N-1}\right).$$

Combining these, we can find the first two terms in the expansion of the logarithmic generator:

$$\mathcal{G}^{(\mathrm{XXX},0)} = N\ln\left(\mathrm{i}\right) + \ln\left(\mathbb{J}_{12}\mathbb{J}_{23}...\mathbb{J}_{N-1,N-2}\mathbb{J}_{N-1,N}\right),$$

$$\mathcal{G}^{(\mathrm{XXX},1)} = (\mathfrak{t}^{(\mathrm{XXX},0)})^{-1}\mathfrak{t}^{(\mathrm{XXX},1)} = -\mathrm{i}\left(\sum_{n=2}^{N}\mathbb{J}_{n,n-1} + \mathbb{J}_{1N}\right), \tag{4.2.11}$$

the second of which can be identified as the XXX Hamiltonian, (4.2.5), after inserting $\mathbb{J}_{nm}$ from (4.2.4) and multiplying by a factor of $\frac{1}{2}$. The total momentum for the system can also be recognised as $\mathcal{G}^{(\mathrm{XXX},0)}$ after it is also multiplied by $\frac{\mathrm{i}}{2}$.

Quantum Ablowitz-Ladik

The qAL model has Lax matrix [26][1]:

$$L_n^{(\text{qAL})}(\lambda) = v_n \begin{pmatrix} e^\lambda & b_n \\ a_n & -e^{-\lambda} \end{pmatrix}, \tag{4.2.12}$$

where the three fields are a_n, b_n, and v_n. The qAL R-matrix is [35]:

$$R^{(\text{qAL})}(\lambda) = \begin{pmatrix} qe^\lambda - q^{-1}e^{-\lambda} & 0 & 0 & 0 \\ 0 & \gamma(e^\lambda - e^{-\lambda}) & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \gamma^{-1}(e^\lambda - e^{-\lambda}) & 0 \\ 0 & 0 & 0 & qe^\lambda - q^{-1}e^{-\lambda} \end{pmatrix}, \tag{4.2.13}$$

where q is some additional parameter of the model, and $\gamma = 1$. These give rise to the following commutation relations:

$$[a_n, v_m] = (1 - q)a_n v_n \delta_{nm}, \qquad [b_n, v_m] = (1 - q^{-1})b_n v_n \delta_{nm},$$
$$\tag{4.2.14}$$
$$[a_n, b_m] = (q - q^{-1})v_n^{-2}\delta_{nm},$$

with Casimir element $qv_n^{-2} - a_n b_n = 1$. The Hamiltonian for this system is:

$$H^{(\text{qAL})} = \sum_{n=1}^{N}(qb_{n+1}a_n + q^{-1}a_{n+1}b_n), \tag{4.2.15}$$

and the evolution equations are:

$$\partial_t a_n = \mathrm{i}(q^{-1} - q)v_n^{-2}(q^{-1}a_{n+1} + qa_{n-1}),$$

$$\partial_t b_n = \mathrm{i}(q - q^{-1})v_n^{-2}(qb_{n+1} + q^{-1}b_{n-1}), \tag{4.2.16}$$

$$\partial_t v_n = \mathrm{i}(q^{-1} - 1)v_n a_n(qb_{n+1} + q^{-1}b_{n-1}) + \mathrm{i}(q - 1)v_n b_n(q^{-1}a_{n+1} + qa_{n-1}).$$

---

[1]The quantum Lax matrix used in this chapter differs from the classical one in Chapter 2 by the factor of -1 in the bottom-right element. Both choices lead to suitable Lax matrices for the Ablowitz-Ladik model, however the removal of the minus sign changes the R-matrix so that $\gamma = q^{-3}$. The classical one is chosen without the minus sign so as to make the connection to the NLS Lax pair through the continuum limit more evident, while the minus sign is kept in the quantum case to keep the chapter in line with the results of [14].

The first step in deriving the commuting quantities for this model is expanding the monodromy matrix:

$$
T^{(\text{qAL})} = v_N...v_1 \left[e^{N\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + e^{(N-1)\lambda} \begin{pmatrix} 0 & b_1 \\ a_N & 0 \end{pmatrix} \right.
$$

$$
+ e^{(N-2)\lambda} T^{(\text{qAL},N-2)} + ... + (-1)^N e^{(2-N)\lambda} T^{(\text{qAL},2-N)} \tag{4.2.17}
$$

$$
\left. -(-1)^N e^{(1-N)\lambda} \begin{pmatrix} 0 & b_N \\ a_1 & 0 \end{pmatrix} + (-1)^N e^{-N\lambda} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right],
$$

where:

$$
T^{(\text{qAL},N-2)} = \begin{pmatrix} \sum_{n=1}^{N-1} b_{n+1}a_n & 0 \\ 0 & a_N b_1 \end{pmatrix},
$$

$$
\tag{4.2.18}
$$

$$
T^{(\text{qAL},2-N)} = \begin{pmatrix} b_N a_1 & 0 \\ 0 & \sum_{n=1}^{N-1} a_{n+1}b_n \end{pmatrix}.
$$

Using these and the fact that $[a_n, b_m]$ is zero when $n \neq m$, the transfer matrix can be determined:

$$
t^{(\text{qAL})} = v_N...v_1 \left[e^{N\lambda} + e^{(N-2)\lambda} \sum_{n=1}^{N} b_{n+1}a_n + ... \right.
$$

$$
\tag{4.2.19}
$$

$$
\left. +(-1)^N e^{(2-N)\lambda} \sum_{n=1}^{N} a_{n+1}b_n + (-1)^N e^{-N\lambda} \right],
$$

which provides a series of commuting conserved quantities. We, however, are interested in the quantities generated by the logarithmic generator $\mathcal{G}(\lambda)$. In order to evaluate this we need to choose one of $\lambda \to \pm\infty$ as the limit. Labelling the coefficient of $e^{\mp k\lambda}$ in the $\lambda \to \pm\infty$ limit as $\mathcal{G}^{(\text{qAL},\mp k)}$, the commuting quantities at the lowest order at either end agree:

$$
\mathcal{G}^{(\text{qAL},0)} = \mathcal{G}^{(\text{qAL},-0)} = \sum_{n=1}^{N} \ln(v_n), \tag{4.2.20}
$$

while they diverge at higher orders:

$$\mathcal{G}^{(\text{qAL},-2)} = \sum_{n=1}^{N} b_{n+1} a_n, \qquad\qquad \mathcal{G}^{(\text{qAL},2)} = \sum_{n=1}^{N} a_{n+1} b_n,$$

$$\mathcal{G}^{(\text{qAL},-4)} = \frac{1}{2} \sum_{n=1}^{N} \left(b_{n+1}(a_n b_n + b_n a_n - 2) a_{n-1} + b_{n+1}^2 a_n^2 \right), \qquad (4.2.21)$$

$$\mathcal{G}^{(\text{qAL},4)} = \frac{1}{2} \sum_{n=1}^{N} \left(a_{n+1}(a_n b_n + b_n a_n - 2) b_{n-1} + a_{n+1}^2 b_n^2 \right).$$

These should be compared to their classical analogues in (2.5.10). Recalling the Hamiltonian for this model, (4.2.15), we can recognise it as the combination of the order 2 terms, $H^{(\text{qAL})} = q\mathcal{G}^{(\text{qAL},-2)} + q^{-1}\mathcal{G}^{(\text{qAL},2)}$.

4.2.2   The A-Matrices

We now use our quantum STS formula, (4.1.20), to derive the A-matrices associated to the commuting quantities calculated above. For each of the models, the first step is to calculate the B-matrices using (4.1.17).

Heisenberg Spin Chain

The lowest order term in the expansion of the B-matrix generator can be read off from (4.2.2) and (4.2.8):

$$\mathbb{B}_{bn}^{(\text{XXX},0)} = i^N \text{tr}_a \left\{ \mathbb{J}_{aN}...\mathbb{J}_{an}(i\mathcal{P}_{ab} - \lambda\mathbb{I})\mathbb{J}_{a,n-1}...\mathbb{J}_{a1} \right\}$$

$$= i^N \mathbb{J}_{12} ... \mathbb{J}_{n-2,n-1}(i\mathbb{J}_{b,n-1}\mathbb{J}_{bn} - \lambda\mathbb{J}_{n-1,n}\mathbb{I})\mathbb{J}_{n,n+1} ... \mathbb{J}_{N-1,N}, \qquad (4.2.22)$$

where $\mathcal{P}_{ab}$ is the permutation matrix and can be read off from (4.2.2). The λ in the R-matrix has a different sign to (4.2.2) because we are considering the lowest order term in the expansion of $R(\mu - \lambda)$ about powers of μ.

Because the total momentum of the XXX spin chain was $\mathcal{G}^{(\text{XXX},0)}$, we can use the zeroth order term from the expansion of the logarithmic A-matrix generator,

(4.1.20), to provide the A-matrix associated to the total momentum:

$$\mathbb{A}_{an}^{(\mathrm{XXX},0)} = \mathrm{i}\mathcal{G}^{(\mathrm{XXX},0)}\mathbb{I} - \mathrm{i}\ln\left(\mathbb{B}_{an}^{(\mathrm{XXX},0)}\right)$$

$$= -\mathrm{i}\ln\left(-1-\lambda^2\right)\mathbb{I} + \mathrm{i}\ln\left(\lambda\mathbb{I} + \mathrm{i}\mathbb{J}_{an}\right),$$

(4.2.23)

which, after removing the overall constant commuting factor is just:

$$\mathbb{A}_{an}^{(\mathrm{XXX},0)} = \mathrm{i}\ln\left(L_{an}\right).$$

(4.2.24)

The A-matrix that we are primarily interested in finding, however, is the one at order λ^1, which will be associated to the Hamiltonian (4.2.5). To find this, we need both the inverse of (4.2.22):

$$(\mathbb{B}_{bn}^{(\mathrm{XXX},0)})^{-1} = \frac{-1}{\mathrm{i}^N(1+\lambda^2)}\mathbb{J}_{N,N-1}\cdots\mathbb{J}_{n+1,n}(\mathrm{i}\mathbb{J}_{bn}\mathbb{J}_{b,n-1} + \lambda\mathbb{J}_{n-1,n}\mathbb{I})\mathbb{J}_{n-1,n-2}\cdots\mathbb{J}_{21},$$

(4.2.25)

and the order λ^1 term in the expansion of the B-matrix generator:

$$\mathbb{B}_{bn}^{(\mathrm{XXX},1)} = \left(\mathfrak{t}^{(\mathrm{XXX},0)} - \lambda\mathfrak{t}^{(\mathrm{XXX},1)}\right)\mathbb{I} + \mathrm{i}^N\Bigg(\mathbb{J}_{23}\cdots\mathbb{J}_{n-1,b}\mathbb{J}_{bn}\cdots\mathbb{J}_{N-1,N}$$

$$+ \sum_{m=2}^{n-2}\mathbb{J}_{12}\cdots\mathbb{J}_{m-1,m+1}\cdots\mathbb{J}_{n-1,b}\mathbb{J}_{bn}\cdots\mathbb{J}_{N-1,N}$$

$$+ \mathbb{J}_{12}\cdots\mathbb{J}_{n-2,b}\mathbb{J}_{bn}\cdots\mathbb{J}_{N-1,N} + \mathbb{J}_{12}\cdots\mathbb{J}_{n-1,b}\mathbb{J}_{b,n+1}\cdots\mathbb{J}_{N-1,N}$$

(4.2.26)

$$+ \sum_{m=n+1}^{N-1}\mathbb{J}_{12}\cdots\mathbb{J}_{n-1,b}\mathbb{J}_{bn}\cdots\mathbb{J}_{m-1,m+1}\cdots\mathbb{J}_{N-1,N}$$

$$+ \mathbb{J}_{12}\cdots\mathbb{J}_{n-1,b}\mathbb{J}_{bn}\cdots\mathbb{J}_{N-2,N-1}\Bigg).$$

Combining these, we can find the next A-matrix in the XXX hierarchy:

$$\mathbb{A}_{bn}^{(\mathrm{XXX},1)} = \mathrm{i}\mathcal{G}^{(\mathrm{XXX},1)}\mathbb{I} - \mathrm{i}(\mathbb{B}_{bn}^{(\mathrm{XXX},0)})^{-1}\mathbb{B}_{bn}^{(\mathrm{XXX},1)}$$

$$= \frac{\mathrm{i}\lambda}{1+\lambda^2}\mathbb{I} + \frac{\mathrm{i}\lambda}{1+\lambda^2}[\mathbb{J}_{b,n-1},\mathbb{J}_{nb}] - \frac{1}{1+\lambda^2}(\mathbb{J}_{bn} + \mathbb{J}_{b,n-1} - \mathbb{J}_{n,n-1}\mathbb{I}).$$

(4.2.27)

Finally, if we remove the constant $\frac{\mathrm{i}\lambda}{1+\lambda^2}\mathbb{I}$ from this and scale it by the same factor

of $\frac{i}{2}$ that we used for the Hamiltonian, we are left with the A-matrix for the XXX spin chain:

$$A_{bn}^{(\text{XXX})} = \frac{-1}{2(1 + \lambda^2)} \left(\lambda[\mathbb{J}_{b,n-1}, \mathbb{J}_{bn}] + \text{i}(\mathbb{J}_{bn} + \mathbb{J}_{b,n-1} - \mathbb{J}_{n,n-1}\mathbb{I}) \right). \tag{4.2.28}$$

Quantum Ablowitz-Ladik

As with the Hamiltonians, we need to consider the two limits $\lambda \to \pm\infty$ of this model separately. Also like for the Hamiltonians, the odd-ordered terms will all be trivial, due to the odd-ordered contributions from the Lax and R-matrices being anti-diagonal. In the limit as $\lambda \to +\infty$, the first two[2] non-trivial B-matrices are:

$$\mathbb{B}_n^{(\text{qAL},-0)} = \text{e}^{-\lambda}v_N...v_1 \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbb{B}_n^{(\text{qAL},-2)} = v_N...v_1 \begin{pmatrix} \text{e}^{-\lambda}(1-q)b_n a_{n-1} - \text{e}^{\lambda}q^{-1} & (q - q^{-1})b_n \\ (q - q^{-1})a_{n-1} & \text{e}^{-\lambda}(q-1)b_n a_{n-1} - \text{e}^{\lambda} \end{pmatrix} \tag{4.2.29}$$

$$+ \text{e}^{-\lambda}v_N...v_1 \sum_{k=1}^{N} b_{k+1}a_k \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

while the first two non-trivial B-matrices in the $\lambda \to -\infty$ limit are:

$$\mathbb{B}_n^{(\text{qAL},0)} = \text{e}^{\lambda}v_N...v_1 \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix},$$

$$\mathbb{B}_n^{(\text{qAL},2)} = v_N...v_1 \begin{pmatrix} \text{e}^{\lambda}(q^{-1}-1)a_n b_{n-1} - \text{e}^{-\lambda} & (q - q^{-1})b_{n-1} \\ (q - q^{-1})a_n & \text{e}^{\lambda}(1-q^{-1})a_n b_{n-1} - \text{e}^{-\lambda}q \end{pmatrix} \tag{4.2.30}$$

$$+ \text{e}^{\lambda}v_N...v_1 \sum_{k=1}^{N} a_{k+1}b_k \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

The Hamiltonian was found by considering the logarithmic generator $\mathcal{G}$, so the B-matrices we actually need are extracted from these by taking the logarithm. In

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[2]We only provide the first two B-matrices from each end as beyond that the expressions become overly bulky.

the $\lambda \to +\infty$ limit, the resulting A-matrices are:

$$\mathbb{A}_n^{(\text{qAL},-0)} = i\mathcal{G}^{(\text{qAL},-0)}\mathbb{I} - i\ln\left(\mathbb{B}_n^{(\text{qAL},-0)}\right)$$

$$= i\begin{pmatrix} \lambda - \ln(q) & 0 \\ 0 & \lambda \end{pmatrix},$$

(4.2.31)

$$\mathbb{A}_n^{(\text{qAL},-2)} = i\mathcal{G}^{(\text{qAL},-2)}\mathbb{I} - i(\mathbb{B}_n^{(\text{qAL},-0)})^{-1}\mathbb{B}_n^{(\text{qAL},-2)}$$

$$= i\begin{pmatrix} (1 - q^{-1})b_n a_{n-1} + e^{2\lambda}q^{-2} & e^{\lambda}q^{-1}(q^{-1} - q)b_n \\ e^{\lambda}(q^{-1} - q)a_{n-1} & (1 - q)b_n a_{n-1} + e^{2\lambda} \end{pmatrix},$$

and the A-matrices arising in the $\lambda \to -\infty$ limit are:

$$\mathbb{A}_n^{(\text{qAL},0)} = i\mathcal{G}^{(\text{qAL},0)}\mathbb{I} - i\ln\left(\mathbb{B}_n^{(\text{qAL},0)}\right)$$

$$= i\begin{pmatrix} -\lambda & 0 \\ 0 & \ln(q) - \lambda \end{pmatrix},$$

(4.2.32)

$$\mathbb{A}_n^{(\text{qAL},2)} = i\mathcal{G}^{(\text{qAL},2)}\mathbb{I} - i(\mathbb{B}_n^{(\text{qAL},0)})^{-1}\mathbb{B}_n^{(\text{qAL},2)}$$

$$= i\begin{pmatrix} (1 - q^{-1})a_n b_{n-1} + e^{-2\lambda} & e^{-\lambda}(q^{-1} - q)b_{n-1} \\ e^{-\lambda}q(q^{-1} - q)a_n & (1 - q)a_n b_{n-1} + e^{-2\lambda}q^2 \end{pmatrix}.$$

Finally, we take the sum of the order 2 components to find the A-matrix equivalent to the Hamiltonian, (4.2.15):

$$A_n^{(\text{qAL})} = q\mathbb{A}_n^{(\text{qAL},-2)} + q^{-1}\mathbb{A}_n^{(\text{qAL},2)}$$

$$= i\begin{pmatrix} (1 - q^{-1})(q^{-1}a_n b_{n-1} + qb_n a_{n-1}) & (q^{-1} - q)(e^{-\lambda}q^{-1}b_{n-1} + e^{\lambda}b_n) \\ (q^{-1} - q)(e^{-\lambda}a_n + e^{\lambda}qa_{n-1}) & (1 - q)(q^{-1}a_n b_{n-1} + qb_n a_{n-1}) \end{pmatrix}$$

$$+ i(e^{2\lambda} + e^{-2\lambda})\begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

(4.2.33)

This should be compared to its classical analogue in (2.4.4).

4.3   Time Evolution in Open Systems

Of particular interest is when integrable boundary conditions are incorporated into this construction. Due to the similar forms of the (rearranged) quantum algebra, (4.1.6), and its discrete equivalent, (2.4.5), the derivation of (4.1.16), the quantum analogue of (2.5.14), followed an almost identical procedure with the non-commutativity of the fields not appearing in any way (outside of the logarithmic generator). When integrable boundary conditions are included, however, additional complexities will arise, leading to an expression that is quadratic in the R-matrix, where the classical version, (2.6.9), was still linear in r.

The boundary conditions are described by some $K_\pm$-matrices that need to satisfy the reflection equations. The K-matrix at the $n = 0$ boundary needs to satisfy [36, 20]:

$$R_{ab}(\lambda-\mu)K_{-,a}(\lambda)R_{ba}(\lambda+\mu)K_{-,b}(\mu) = K_{-,b}(\mu)R_{ab}(\lambda+\mu)K_{-,a}(\lambda)R_{ba}(\lambda-\mu), \quad (4.3.1)$$

while the K-matrix at the $n = N + 1$ boundary obeys an equivalent relation [37]:

$$\left(R_{ab}^{-1}(\lambda-\mu)\right)^{t_{ab}} K_{+,a}^{t_a}(\lambda)\left(\left(R_{ba}^{t_a}(\lambda+\mu)\right)^{-1}\right)^{t_b} K_{+,b}^{t_b}(\mu)$$

$$(4.3.2)$$

$$= K_{+,b}^{t_b}(\mu)\left(\left(R_{ab}^{t_a}(\lambda+\mu)\right)^{-1}\right)^{t_b} K_{+,a}^{t_a}(\lambda)\left(R_{ba}^{-1}(\lambda-\mu)\right)^{t_{ab}},$$

where ^{t_a}, ^{t_b}, and $^{t_{ab}}$ are partial transpositions in the $_a$, $_b$, and combined spaces, respectively. In the classical limit (assuming that the boundary fields are static) the classical reflection equation used in the prior chapters, (2.3.1), is returned.

Using these boundary matrices, the transfer matrix $\mathfrak{t}$ is replaced with Sklyanin's double row transfer matrix [20]:

$$\bar{\mathfrak{t}}(\lambda) = \mathrm{tr}_a \left\{ K_{+,a}(\lambda)T_a(\lambda)K_{-,a}(\lambda)T_a^{-1}(-\lambda) \right\}. \quad (4.3.3)$$

Before we can show that this commutes with itself for different spectral parameters, we need to first prove that $\mathcal{K}_-(\lambda) = T(\lambda)K_-(\lambda)T^{-1}(-\lambda)$ satisfies the reflection equation, (4.3.1). To do this we make repeated use of the RTT relation, (4.1.8),

which we mark by square brackets [...]:

$$\text{LHS (4.3.1)} = R_{ab}(\lambda - \mu)T_a(\lambda)K_{-,a}(\lambda)\left[T_a^{-1}(-\lambda)R_{ba}(\lambda + \mu)T_b(\mu)\right]K_{-,b}(\mu)T_b^{-1}(-\mu)$$

$$= R_{ab}(\lambda - \mu)T_a(\lambda)K_{-,a}(\lambda)\left[T_b(\mu)R_{ba}(\lambda + \mu)T_a^{-1}(-\lambda)\right]K_{-,b}(\mu)T_b^{-1}(-\mu)$$

$$= \left[R_{ab}(\lambda - \mu)T_a(\lambda)T_b(\mu)\right]K_{-,a}(\lambda)R_{ba}(\lambda + \mu)K_{-,b}(\mu)T_a^{-1}(-\lambda)T_b^{-1}(-\mu)$$

$$= \left[T_b(\mu)T_a(\lambda)R_{ab}(\lambda - \mu)\right]K_{-,a}(\lambda)R_{ba}(\lambda + \mu)K_{-,b}(\mu)T_a^{-1}(-\lambda)T_b^{-1}(-\mu),$$
$$(4.3.4)$$

the central terms of which comprise the reflection equation, (4.3.1). Using (4.3.1) to reverse these terms, this becomes:

$$\ldots = T_b(\mu)T_a(\lambda)K_{-,b}(\mu)R_{ab}(\lambda + \mu)K_{-,a}(\lambda)\left[R_{ba}(\lambda - \mu)T_a^{-1}(-\lambda)T_b^{-1}(-\mu)\right]$$

$$= T_b(\mu)K_{-,b}(\mu)T_a(\lambda)R_{ab}(\lambda + \mu)K_{-,a}(\lambda)\left[T_b^{-1}(-\mu)T_a^{-1}(-\lambda)R_{ba}(\lambda - \mu)\right]$$

$$= T_b(\mu)K_{-,b}(\mu)\left[T_a(\lambda)R_{ab}(\lambda + \mu)T_b^{-1}(-\mu)\right]K_{-,a}(\lambda)T_a^{-1}(-\lambda)R_{ba}(\lambda - \mu) \quad (4.3.5)$$

$$= T_b(\mu)K_{-,b}(\mu)\left[T_b^{-1}(-\mu)R_{ab}(\lambda + \mu)T_a(\lambda)\right]K_{-,a}(\lambda)T_a^{-1}(-\lambda)R_{ba}(\lambda - \mu)$$

$$= \text{RHS (4.3.1)}.$$

Because $\mathcal{K}_-$ is still a solution of (4.3.1), the proof that the open transfer matrices defined in (4.3.3) commute then follows by considering their product in one order (we drop the parameter dependence of the K_+ and $\mathcal{K}_-$ matrices, as it can be implicitly extracted from the subscript, e.g. $K_{+,a}$ and $\mathcal{K}_{-,a}$ are functions of λ) [20, 37] (for a diagrammatic approach, see [38]):

$$\bar{\mathfrak{t}}(\lambda)\bar{\mathfrak{t}}(\mu) = \text{tr}_a\left\{K_{+,a}^{t_a}(\lambda)\mathcal{K}_{-,a}^{t_a}(\lambda)\right\}\text{tr}_b\left\{K_{+,b}(\mu)\mathcal{K}_{-,b}(\mu)\right\}$$

$$= \text{tr}_{ab}\left\{K_{+,a}^{t_a}K_{+,b}\mathcal{K}_{-,a}^{t_a}\mathcal{K}_{-,b}\right\}$$
$$(4.3.6)$$

$$= \text{tr}_{ab}\left\{K_{+,a}^{t_a}K_{+,b}\left(R_{ba}^{t_a}(\lambda + \mu)\right)^{-1}R_{ba}^{t_a}(\lambda + \mu)\mathcal{K}_{-,a}^{t_a}\mathcal{K}_{-,b}\right\}$$

$$= \text{tr}_{ab}\left\{\left(K_{+,a}^{t_a}\left(\left(R_{ba}^{t_a}(\lambda + \mu)\right)^{-1}\right)^{t_b}K_{+,b}^{t_b}\right)^{t_b}\left(\mathcal{K}_{-,a}R_{ba}(\lambda + \mu)\mathcal{K}_{-,b}\right)^{t_a}\right\}.$$

To continue, we make use of the identity $\mathrm{tr}_a\{A_{ab}B_{ab}\} = \mathrm{tr}_a\{A_{ab}^{t_a}B_{ab}^{t_a}\}$ between partial traces and partial transpositions. After using this to move the ^{t_a} from the second bracket to the first, this is:

$$\bar{\mathfrak{t}}(\lambda)\bar{\mathfrak{t}}(\mu) = \mathrm{tr}_{ab}\left\{ \left(K_{+,a}^{t_a}\left(\left(R_{ba}^{t_a}(\lambda+\mu)\right)^{-1}\right)^{t_b} K_{+,b}^{t_b}\right)^{t_{ab}} \left(\mathcal{K}_{-,a}R_{ba}(\lambda+\mu)\mathcal{K}_{-,b}\right)\right\}$$

$$= \mathrm{tr}_{ab}\left\{ \left(K_{+,a}^{t_a}\left(\left(R_{ba}^{t_a}(\lambda+\mu)\right)^{-1}\right)^{t_b} K_{+,b}^{t_b}\right)^{t_{ab}} R_{ab}^{-1}(\lambda-\mu)\right.$$

$$\left. \times R_{ab}(\lambda-\mu)\left(\mathcal{K}_{-,a}R_{ba}(\lambda+\mu)\mathcal{K}_{-,b}\right)\right\} \tag{4.3.7}$$

$$= \mathrm{tr}_{ab}\left\{ \left(\left(R_{ab}^{-1}(\lambda-\mu)\right)^{t_{ab}} K_{+,a}^{t_a}\left(\left(R_{ba}^{t_a}(\lambda+\mu)\right)^{-1}\right)^{t_b} K_{+,b}^{t_b}\right)^{t_{ab}}\right.$$

$$\left. \times \left(R_{ab}(\lambda-\mu)\mathcal{K}_{-,a}R_{ba}(\lambda+\mu)\mathcal{K}_{-,b}\right)\right\}.$$

The terms inside each of the brackets can be recognised as the two reflection equations, (4.3.1) and (4.3.2). After the application of these reflection equations we can perform the same steps as above, but in reverse order, cancelling the factors of R and R^{-1} by cycling them around the outside of the trace, to complete the proof. Then, as the generators commute, so too will the coefficients in their power series, labelled $\bar{\mathfrak{t}}^{(k)}$:

$$\left[\bar{\mathfrak{t}}^{(k)}, \bar{\mathfrak{t}}^{(j)}\right] = 0. \tag{4.3.8}$$

To extract local quantities from this, we again consider the logarithm of $\bar{\mathfrak{t}}$, labelled $\bar{\mathcal{G}}$. The coefficients $\bar{\mathcal{G}}^{(k)}$ in the expansion of $\bar{\mathcal{G}}(\lambda)$ then also commute with one another:

$$\left[\bar{\mathcal{G}}^{(k)}, \bar{\mathcal{G}}^{(j)}\right] = 0. \tag{4.3.9}$$

Our goal is to derive a generator for the open A-matrices, labelled $\bar{\mathbb{A}}_n$, corresponding to each of the Hamiltonians generated by $\bar{\mathfrak{t}}(\lambda)$ or $\bar{\mathcal{G}}(\lambda)$. We consider the non-logarithmic version first. Just as when deriving the version with periodic boundary conditions, we start by considering the time evolution of L_n along some

master time-flow that incorporates all of the time-flows corresponding to each of the Hamiltonians:

$$\dot{L}_{bn}(\mu) = \mathrm{i}\big[\bar{\mathfrak{t}}(\lambda), L_{bn}(\mu)\big]$$

$$= \mathrm{itr}_a\left\{ K_{+,a}T_a(N, n+1)\big[L_{an}(\lambda), L_{bn}(\mu)\big]T_a(n-1, 1)K_{-,a}T_a^{-1} \right. \tag{4.3.10}$$

$$\left. + K_{+,a}T_aK_{-,a}T_a^{-1}(n-1, 1)\big[L_{an}^{-1}(-\lambda), L_{bn}(\mu)\big]T_a^{-1}(N, n+1) \right\}.$$

In order to evaluate the second commutator we need to have an expression for the commutator with an inverse, which is:

$$\big[L_{an}^{-1}(-\lambda), L_{bm}(\mu)\big] = \big(L_{an}^{-1}(-\lambda)\tilde{\mathfrak{R}}_{ab}(\lambda+\mu)L_{bm}(\mu) - L_{bm}(\mu)\tilde{\mathfrak{R}}_{ab}(\lambda+\mu)L_{an}^{-1}(-\lambda)\big)\delta_{nm},$$
$$\tag{4.3.11}$$

where $\tilde{\mathfrak{R}}_{ab}(\lambda) = \mathbb{I} - R_{ab}^{-1}(-\lambda)$. With this, the expression above becomes:

$$\dot{L}_{bn}(\mu) = \mathrm{itr}_a\left\{ K_{+,a}T_aK_{-,a}T_a^{-1}(n, 1)\tilde{\mathfrak{R}}_{ab}(\lambda+\mu)T_a^{-1}(N, n+1) \right\} L_{bn}$$

$$- \mathrm{i}L_{bn}\mathrm{tr}_a\left\{ K_{+,a}T_a(N, n)\mathfrak{R}_{ab}(\lambda-\mu)T_a(n-1, 1)K_{-,a}T_a^{-1} \right\}$$

$$+ \mathrm{itr}_a\left\{ K_{+,a}T_a(N, n+1)\mathfrak{R}_{ab}(\lambda-\mu)T_a(n, 1)L_{bn}K_{-,a}T_a^{-1} \right\} \tag{4.3.12}$$

$$- \mathrm{itr}_a\left\{ K_{+,a}T_aK_{-,a}L_{bn}T_a^{-1}(n-1, 1)\tilde{\mathfrak{R}}_{ab}(\lambda+\mu)T_a^{-1}(N, n) \right\}.$$

While the first two terms here are reminiscent of the zero-curvature condition, the latter two are not, as they contain an L_{bn} that is "trapped" in the middle of each of them. If this were the classical case, we could simply commute it out through the T_a or T_a^{-1}, however, due to the non-commutativity of the fields that cannot be done here. Instead we use the two algebraic expressions between the L-matrices, (4.1.6) and (4.3.11), to commute the inner terms out. Obviously, when we do this we will pick up another term that is still "trapped" between the two $\mathfrak{R}$-matrices, but as we are doing this with both the third and fourth terms in the above expression we get two such contributions, which cancel out with one another. The resulting right-hand side then takes the form of a zero-curvature condition:

$$\dot{L}_{bn}(\mu) = \bar{\mathbb{A}}_{bn}(\lambda, \mu)L_{bm}(\mu) - L_{bn}(\mu)\bar{\mathbb{A}}_{bn}(\lambda, \mu), \tag{4.3.13}$$

where we define the generator of the A-matrices in the context of open boundary conditions as:

$$\bar{\mathbb{A}}_{bn}(\lambda, \mu) = \mathrm{itr}_a \left\{ K_{+,a} T_a(N, n) \mathfrak{R}_{ab}(\lambda - \mu) T_a(n-1, 1) K_{-,a} T_a^{-1} \right\}$$

$$+ \mathrm{itr}_a \left\{ K_{+,a} T_a K_{-,a} T_a^{-1}(n-1, 1) \tilde{\mathfrak{R}}_{ab}(\lambda + \mu) T_a^{-1}(N, n) \right\}$$

$$- \mathrm{itr}_a \left\{ K_{+,a} T_a(N, n) \mathfrak{R}_{ab}(\lambda - \mu) T_a(n-1, 1) \right.$$

$$\left. \times K_{-,a} T_a^{-1}(n-1, 1) \tilde{\mathfrak{R}}_{ab}(\lambda + \mu) T_a^{-1}(N, n) \right\}, \tag{4.3.14}$$

or after expanding the $\mathfrak{R}_{ab}(\lambda) = \mathbb{I} - R_{ab}(\lambda)$ and the $\tilde{\mathfrak{R}}_{ab}(\lambda) = \mathbb{I} - R_{ab}^{-1}(-\lambda)$, this becomes [14]:

$$\bar{\mathbb{A}}_{bn}(\lambda, \mu) = \mathrm{i}\bar{\mathfrak{t}}(\lambda)\mathbb{I} - \mathrm{i}\bar{\mathbb{B}}_{bn}(\lambda, \mu), \tag{4.3.15}$$

where we define the open equivalent of (4.1.17):

$$\bar{\mathbb{B}}_{bn}(\lambda, \mu) = \mathrm{tr}_a \left\{ K_{+,a}(\lambda) T_a(N, n; \lambda) R_{ab}(\lambda - \mu) T_a(n-1, 1; \lambda) \right.$$

$$\left. \times K_{-,a}(\lambda) T_a^{-1}(n-1, 1; -\lambda) R_{ab}^{-1}(-\lambda - \mu) T_a^{-1}(N, n; -\lambda) \right\}. \tag{4.3.16}$$

It is important to note here that, compared to the closed version (4.1.17), this has lost the term linear in R and instead gained a term quadratic in R. This is because all of the information from the underlying algebra needs to be concealed within this expression, which, in the closed case was just the commutation relations extracted from the RLL relation, (4.1.5). In this open case on the other hand, the reflection equations (which are quadratic in R) also need to be encoded. This is in stark contrast to the classical results, which are both linear, but this is because they are instead reflecting the r-dependence of the classical Poisson brackets, (2.4.5), and the classical reflection equation, (2.3.1), both of which are linear in r.

At each of the boundaries we make use of how $T(a, b) = \mathbb{I}$ for $b > a$ to find the two boundary B-matrix generators, which are used in (4.3.15) to find the boundary

A-matrix generators. At the $n = N + 1$ boundary, the generator is:

$$\bar{\mathbb{B}}_{b,N+1}(\lambda, \mu) = \text{tr}_a \left\{ K_{+,a}(\lambda) R_{ab}(\lambda - \mu) T_a(\lambda) K_{-,a}(\lambda) T_a^{-1}(-\lambda) R_{ab}^{-1}(-\lambda - \mu) \right\},$$

(4.3.17)

while at the $n = 1$ boundary it is instead:

$$\bar{\mathbb{B}}_{b1}(\lambda, \mu) = \text{tr}_a \left\{ K_{+,a}(\lambda) T_a(\lambda) R_{ab}(\lambda - \mu) K_{-,a}(\lambda) R_{ab}^{-1}(-\lambda - \mu) T_a^{-1}(-\lambda) \right\}. \quad (4.3.18)$$

Logarithmic Generator

Because we are interested in the local quantities extracted from the logarithmic generator $\mathcal{G}(\lambda)$ we need to find a logarithmic analogue of (4.3.15). Just as the non-commutativity was a potential problem in the closed case, so too must we be careful here, and so we again simply define it using the logarithm of $\bar{\mathbb{B}}_{bn}$:

$$\bar{\mathbb{A}}_{bn}(\lambda, \mu) = \mathrm{i}\bar{\mathcal{G}}(\lambda)\mathbb{I} - \mathrm{i}\ln\left(\bar{\mathbb{B}}_{bn}(\lambda, \mu)\right). \quad (4.3.19)$$

As we shall see this provides the correct classical limit. In order to prove that this logarithm is well defined (that is, that the different $\bar{\mathbb{B}}_{bn}^{(k)}$ commute with one another), we define a term in analogy to the proof in (4.3.4):

$$\mathcal{K}_{-,ab}(\lambda, \mu) = T_a(N, n; \lambda) R_{ab}(\lambda - \mu) T_a(n - 1, 1; \lambda) K_{-,a}(\lambda)$$

(4.3.20)

$$\times T_a^{-1}(n - 1, 1; -\lambda) R_{ab}^{-1}(-\lambda - \mu) T_a^{-1}(N, n; -\lambda),$$

which can be shown to be a solution of the reflection equation, (4.3.1), through the same method as in (4.3.4), except with additional applications of the Yang-Baxter equation, (4.1.3). The task then becomes proving that:

$$\left[\text{tr}_a \left\{ K_{+,a}(\mu) \mathcal{K}_{-,ac}(\mu, \lambda) \right\}, \text{tr}_b \left\{ K_{+,b}(\xi) \mathcal{K}_{-,bc}(\xi, \lambda) \right\} \right] = 0, \quad (4.3.21)$$

which is simply a repeat of the fact that the open transfer matrices commute, as proven in (4.3.6).

Classical Limits

The classical limit we consider for these two generators is the same as from Section 4.1, $R_{ab}(\lambda) \to \mathbb{I} + i\hbar r_{ab}(\lambda) + ...$ (which also implies that $R_{ab}^{-1}(\lambda) \to \mathbb{I} - i\hbar r_{ab}(\lambda) + ...$) and $L_{an} \to \hbar L_{an}$. Again, the first thing to check is the limit of $\bar{\mathbb{B}}_{bn}$:

$$\bar{\mathbb{B}}_{bn}(\lambda, \mu) \to \bar{\mathfrak{t}}(\lambda)\mathbb{I} + i\hbar\Big(\mathrm{tr}_a \left\{ K_{+,a}T_a(N,n)r_{ab}(\lambda - \mu)T_a(n-1,1)K_{-,a}T_a^{-1}(-\lambda) \right\}$$

$$- \mathrm{tr}_a \left\{ K_{+,a}T_aK_{-,a}T_a^{-1}(n-1,1; -\lambda)r_{ab}(-\lambda - \mu)T_a^{-1}(N,n; -\lambda) \right\} \Big) + ...,$$

$$(4.3.22)$$

so that the limit of the non-logarithmic A-matrix generator, (4.3.15), is:

$$\bar{\mathbb{A}}_{bn}(\lambda, \mu) \to \hbar\Big(\mathrm{tr}_a \left\{ K_{+,a}T_a(N,n)r_{ab}(\lambda - \mu)T_a(n-1,1)K_{-,a}T_a^{-1}(-\lambda) \right\}$$

$$- \mathrm{tr}_a \left\{ K_{+,a}T_aK_{-,a}T_a^{-1}(n-1,1; -\lambda)r_{ab}(-\lambda - \mu)T_a^{-1}(N,n; -\lambda) \right\} \Big) + ...,$$

$$(4.3.23)$$

which matches the expected result. Similarly for the logarithmic generator, we need to expand the logarithm of this:

$$\bar{\mathbb{A}}_{bn}(\lambda, \mu) \to \hbar\bar{\mathfrak{t}}^{-1}(\lambda)\Big(\mathrm{tr}_a \left\{ K_{+,a}T_a(N,n)r_{ab}(\lambda - \mu)T_a(n-1,1)K_{-,a}T_a^{-1}(-\lambda) \right\}$$

$$- \mathrm{tr}_a \left\{ K_{+,a}T_aK_{-,a}T_a^{-1}(n-1,1; -\lambda)r_{ab}(-\lambda - \mu)T_a^{-1}(N,n; -\lambda) \right\} \Big) + ...,$$

$$(4.3.24)$$

which is also in line with our expectations of (2.6.9).

4.4 Examples of Open A-Matrices

Heisenberg Spin Chain

We start with the example of the Heisenberg spin chain again. For the XXX model[3], the general K-matrix solutions are [22]:

$$
K_-^{(\text{XXX})}(\lambda) = \begin{pmatrix} \alpha_- + \lambda\delta_- & \lambda\beta_- \\ \lambda\gamma_- & \alpha_- - \lambda\delta_- \end{pmatrix},
$$

$$
K_+^{(\text{XXX})}(\lambda) = \begin{pmatrix} \alpha_+ + \mathrm{i}\delta_+ + \lambda\delta_+ & \mathrm{i}\beta_+ + \lambda\beta_+ \\ \mathrm{i}\gamma_+ + \lambda\gamma_+ & \alpha_+ - \mathrm{i}\delta_+ - \lambda\delta_+ \end{pmatrix}.
$$

(4.4.1)

The first two terms in the expansion of the double-row transfer matrix are:

$$
\bar{\mathfrak{t}}^{(\text{XXX},0)} = 2\mathrm{i}^{2N}\alpha_+\alpha_-,
$$

$$
\bar{\mathfrak{t}}^{(\text{XXX},1)} = 4\mathrm{i}^{2N-1}\alpha_+\alpha_- \left(\sum_{m=1}^{N-1} \mathbb{J}_{m,m+1} + \frac{1}{2} \right) + 2\mathrm{i}^{2N}\alpha_- \left(\beta_+ J_N^+ + \gamma_+ J_N^- + 2\delta_+ J_N^z \right)
$$

$$
+ 2\mathrm{i}^{2N}\alpha_+ \left(\beta_- J_1^+ + \gamma_- J_1^- + 2\delta_- J_1^z \right),
$$

(4.4.2)

which give the local Hamiltonians through the use of the logarithmic generator:

$$
\bar{\mathcal{G}}^{(\text{XXX},0)} = \ln\left(2\mathrm{i}^{2N}\alpha_+\alpha_- \right),
$$

$$
\bar{\mathcal{G}}^{(\text{XXX},1)} = -2\mathrm{i} \left(\sum_{m=1}^{N-1} \mathbb{J}_{m,m+1} + \frac{1}{2} \right) + \frac{\beta_+}{\alpha_+} J_N^+ + \frac{\gamma_+}{\alpha_+} J_N^- + 2\frac{\delta_+}{\alpha_+} J_N^z
$$

(4.4.3)

$$
+ \frac{\beta_-}{\alpha_-} J_1^+ + \frac{\gamma_-}{\alpha_-} J_1^- + 2\frac{\delta_-}{\alpha_-} J_1^z.
$$

The bulk contribution from the open Hamiltonian, $\bar{H}^{(\text{XXX})} = \frac{\mathrm{i}}{2}\bar{\mathcal{G}}^{(\text{XXX},1)}$, is merely double the periodic Hamiltonian, as expected. Consequently, the bulk equations of motion will still be (4.2.6). At the boundaries, however, we pick up contributions

---

[3]The XXX R-matrix obeys a condition called "crossing unitarity", which states that there exists an η such that $R_{ab}^{t_a}(\lambda)R_{ab}^{t_a}(-\lambda - \eta) \propto \mathbb{I}$. When this is the case, the two K-matrices reduce to the same solution, but with the spectral parameter in K_+ shifted by a factor of η. For this model, the choice of η is $\eta = \mathrm{i}$.

from the boundary fields, so that at $n = N$, they are:

$$\partial_t J_N^z = iJ_N^- \left(\frac{i\gamma_+}{2\alpha_+} + J_{N-1}^+ \right) - iJ_N^+ \left(\frac{i\beta_+}{2\alpha_+} + J_{N-1}^- \right),$$

$$\partial_t J_N^+ = 2iJ_N^+ \left(\frac{i\delta_+}{2\alpha_+} + J_{N-1}^z \right) - 2iJ_N^z \left(\frac{i\gamma_+}{2\alpha_+} + J_{N-1}^+ \right), \qquad (4.4.4)$$

$$\partial_t J_N^- = -2iJ_N^- \left(\frac{i\delta_+}{2\alpha_+} + J_{N-1}^z \right) + 2iJ_N^z \left(\frac{i\beta_+}{2\alpha_+} + J_{N-1}^- \right),$$

while at the $n = 1$ boundary they are:

$$\partial_t J_1^z = iJ_1^- \left(J_2^+ + \frac{i\gamma_-}{2\alpha_-} \right) - iJ_1^+ \left(J_2^- + \frac{i\beta_-}{2\alpha_-} \right),$$

$$\partial_t J_1^+ = 2iJ_1^+ \left(J_2^z + \frac{i\delta_-}{2\alpha_-} \right) - 2iJ_1^z \left(J_2^+ + \frac{i\gamma_-}{2\alpha_-} \right), \qquad (4.4.5)$$

$$\partial_t J_1^- = -2iJ_1^- \left(J_2^z + \frac{i\delta_-}{2\alpha_-} \right) + 2iJ_1^z \left(J_2^- + \frac{i\beta_-}{2\alpha_-} \right).$$

We now turn to deriving the A-matrices associated to the system at its boundaries. Just as the Hamiltonian is the same in the bulk, the A-matrices in the bulk agree with (4.2.28). Consequently, we only discuss here the boundary matrices, $A_{N+1}^{(XXX)}$ and $A_1^{(XXX)}$, which first need the B-matrices found using (4.3.17) and (4.3.18). It will be useful to write the K-matrices more compactly by defining:

$$\mathbb{K}_\pm = \frac{i}{2\alpha_\pm} \begin{pmatrix} \delta_\pm & \beta_\pm \\ \gamma_\pm & -\delta_\pm \end{pmatrix}, \qquad (4.4.6)$$

so that the B-matrices are:

$$\bar{\mathbb{B}}_{N+1}^{(XXX,0)} = \bar{\mathbb{B}}_1^{(XXX,0)} = -2(1+\lambda^2)i^{2N}\alpha_+\alpha_-\mathbb{I},$$

$$\bar{\mathbb{B}}_{N+1}^{(XXX,1)} = \bar{\mathbb{B}}_{N+1}^{(XXX,0)} \left(\bar{\mathcal{G}}^{(XXX,1)} - \frac{2}{1+\lambda^2} \left(\lambda\left[\mathbb{J}_N, \mathbb{K}_+\right] + i(\mathbb{K}_+ + \mathbb{J}_N - \mathbb{J}_N\mathbb{K}_+\mathbb{J}_N) \right) \right),$$

$$\bar{\mathbb{B}}_1^{(XXX,1)} = \bar{\mathbb{B}}_1^{(XXX,0)} \left(\bar{\mathcal{G}}^{(XXX,1)} - \frac{2}{1+\lambda^2} \left(\lambda[\mathbb{K}_-, \mathbb{J}_1] + i(\mathbb{J}_1 + \mathbb{K}_- - \mathbb{J}_1\mathbb{K}_-\mathbb{J}_1) \right) \right).$$

$$(4.4.7)$$

Using these with (4.1.20), we can now derive the A-matrices for each of these

boundary cases:

$$\bar{\mathbb{A}}_{N+1}^{(\text{XXX},0)} = \bar{\mathbb{A}}_1^{(\text{XXX},0)} = -\mathrm{i}\ln\left(-1 - \lambda^2\right)\mathbb{I},$$

$$\bar{\mathbb{A}}_{N+1}^{(\text{XXX},1)} = \frac{2\mathrm{i}}{1+\lambda^2}\left(\lambda\left[\mathbb{J}_N, \mathbb{K}_+\right] + \mathrm{i}(\mathbb{K}_+ + \mathbb{J}_N - \mathbb{J}_N\mathbb{K}_+\mathbb{J}_N)\right), \qquad (4.4.8)$$

$$\bar{\mathbb{A}}_1^{(\text{XXX},1)} = \frac{2\mathrm{i}}{1+\lambda^2}\left(\lambda[\mathbb{K}_-, \mathbb{J}_1] + \mathrm{i}(\mathbb{J}_1 + \mathbb{K}_- - \mathbb{J}_1\mathbb{K}_-\mathbb{J}_1)\right),$$

so that the A-matrices for the XXX model at the boundaries are found by scaling these by $\frac{\mathrm{i}}{2}$:

$$\bar{A}_{N+1}^{(\text{XXX})} = \frac{-1}{1+\lambda^2}\left(\lambda\left[\mathbb{J}_N, \mathbb{K}_+\right] + \mathrm{i}(\mathbb{K}_+ + \mathbb{J}_N - \mathbb{J}_N\mathbb{K}_+\mathbb{J}_N)\right),$$

$$\qquad\qquad (4.4.9)$$

$$\bar{A}_1^{(\text{XXX})} = \frac{-1}{1+\lambda^2}\left(\lambda[\mathbb{K}_-, \mathbb{J}_1] + \mathrm{i}(\mathbb{J}_1 + \mathbb{K}_- - \mathbb{J}_1\mathbb{K}_-\mathbb{J}_1)\right).$$

By comparing these with the periodic A-matrix, (4.2.28), we can see that the introduction of open boundary conditions to the model only manifests as replacing $\mathbb{J}_{N+1}$ and $\mathbb{J}_0$ (which due to the periodic boundary conditions were identified with $\mathbb{J}_1$ and $\mathbb{J}_N$ respectively) with the constant matrices $\mathbb{K}_+$ and $\mathbb{K}_-$, respectively. The boundary equations, (4.4.4) and (4.4.5), can be related to the bulk equations, (4.2.7), by the same substitution.

Quantum Ablowitz-Ladik

When considering the periodic qAL model we needed to consider both limits $\lambda \to \pm\infty$ in order to find the Hamiltonian and A-matrix. In the case of open boundary conditions however, the results from the two limits will be the same, so we need only consider one of them. We choose the $\lambda \to +\infty$ limit.

The K-matrices extracted from (4.3.1) and (4.3.2) with the qAL R-matrix[4] are[5]:

$$K_-^{(\mathrm{qAL})}(\lambda) = \begin{pmatrix} \mathrm{e}^\lambda + \alpha_-\mathrm{e}^{-\lambda} & 0 \\ 0 & \alpha_-\mathrm{e}^\lambda + \mathrm{e}^{-\lambda} \end{pmatrix},$$

$$K_+^{(\mathrm{qAL})}(\lambda) = \begin{pmatrix} q^2\mathrm{e}^\lambda + q^{-2}\alpha_+\mathrm{e}^{-\lambda} & 0 \\ 0 & q^2\alpha_+\mathrm{e}^\lambda + q^{-2}\mathrm{e}^{-\lambda} \end{pmatrix}.$$

(4.4.10)

Before we can expand $\bar{\mathfrak{t}}(\lambda)$ and $\bar{\mathcal{G}}(\lambda)$ we first need to find the inverse of $L_{an}(\lambda)$ for calculating $T_a^{-1}(-\lambda)$:

$$L_{an}^{-1}(-\lambda) = v_n \begin{pmatrix} q\mathrm{e}^\lambda & b_n \\ a_n & -q^{-1}\mathrm{e}^{-\lambda} \end{pmatrix}. \qquad (4.4.11)$$

Using these, the first two non-trivial terms in the expansion of the transfer matrix in the limit as $\lambda \to \infty$ are (like in the periodic case, the terms with odd powers of λ will all be trivial):

$$\bar{\mathfrak{t}}^{(\mathrm{qAL},0)} = q^{N+2}v_N^2...v_1^2,$$

$$\bar{\mathfrak{t}}^{(\mathrm{qAL},2)} = q^N v_N^2...v_1^2\bigg(q\sum_{n=1}^{N-1}(qb_{n+1}a_n + q^{-1}a_{n+1}b_n) \qquad (4.4.12)$$

$$+ q(q\alpha_-b_1a_1 + q^{-1}\alpha_+a_Nb_N) + (q^{-2}\alpha_+ + q^2\alpha_-)\bigg),$$

so that the corresponding two terms in the expansion of $\bar{\mathcal{G}}(\lambda)$ are:

$$\bar{\mathcal{G}}^{(\mathrm{qAL},0)} = 2\sum_{n=1}^{N}\ln(v_n) + (N+2)\ln(q),$$

$$\bar{\mathcal{G}}^{(\mathrm{qAL},2)} = q^{-1}\sum_{n=1}^{N-1}(qb_{n+1}a_n + q^{-1}a_{n+1}b_n) + q^{-1}(q\alpha_-b_1a_1 + q^{-1}\alpha_+a_Nb_N) \qquad (4.4.13)$$

$$+ (q^{-4}\alpha_+ + \alpha_-).$$

---

[4]Like the XXX R-matrix, this also satisfies crossing unitarity, but with $\eta = 2\ln(q)$.

[5]Note that there are actually two distinct choices of K-matrices, those given in (4.4.10) and a constant anti-diagonal version, $K_\pm(\lambda) = \begin{pmatrix} 0 & 1 \\ \alpha_\pm & 0 \end{pmatrix}$. We choose to only consider the diagonal version to make the comparisons with the classical case (where only the diagonal version exists) apparent.

Removing the constant term from $\bar{\mathcal{G}}^{(\text{qAL},2)}$ and multiplying by a factor of q, this is the open Hamiltonian for the qAL model:

$$\bar{H}^{(\text{qAL})} = \sum_{n=1}^{N-1} (q b_{n+1} a_n + q^{-1} a_{n+1} b_n) + (q \alpha_- b_1 a_1 + q^{-1} \alpha_+ a_N b_N). \tag{4.4.14}$$

This should be compared against the closed Hamiltonian, (4.2.15). In the bulk, the non-periodic boundary conditions have no effect on the Hamiltonian, so we will only provide the evolution equations for the boundary terms.

At $n = N$, the equations of motion take the form [34]:

$$\partial_t a_N = \mathrm{i}(1 - q^2) v_N^{-2} (a_{N-1} + \alpha_+ a_N),$$

$$\partial_t b_N = \mathrm{i}(1 - q^{-2}) v_N^{-2} (b_{N-1} + \alpha_+ b_N), \tag{4.4.15}$$

$$\partial_t v_N = \mathrm{i}\big((q^{-1} - 1) a_N b_{N-1} + (q - 1) b_N a_{N-1}\big) v_N,$$

while at the $n = 1$ boundary, they are:

$$\partial_t a_1 = \mathrm{i}(q^{-2} - 1)(a_2 + \alpha_- a_1) v_1^{-2},$$

$$\partial_t b_1 = \mathrm{i}(q^2 - 1)(b_2 + \alpha_- b_1) v_1^{-2}, \tag{4.4.16}$$

$$\partial_t v_1 = \mathrm{i} v_1 \big((1 - q^{-1}) a_2 b_1 + (1 - q) b_2 a_1\big).$$

We now turn to finding the B-matrices for the open version of this model, with the goal of finding the A-matrix. Due to the bulk A-matrices being the same as their periodic versions, we only derive here the boundary terms, $\bar{A}_{N+1}^{(\text{qAL})}$ and $\bar{A}_1^{(\text{qAL})}$.

The first two non-trivial B-matrices at each boundary are:

$$\bar{\mathbb{B}}_{N+1}^{(\mathrm{qAL},0)} = \bar{\mathbb{B}}_1^{(\mathrm{qAL},0)} = q^{N+2}v_N^2...v_1^2 \begin{pmatrix} q^2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\bar{\mathbb{B}}_{N+1}^{(\mathrm{qAL},2)} = \bar{\mathbb{B}}_{N+1}^{(\mathrm{qAL},0)} \left(\bar{\mathcal{G}}^{(\mathrm{qAL},2)} - (\mathrm{e}^{2\lambda} + \mathrm{e}^{-2\lambda}) \begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix} \right.$$

$$\left. + (1 - q^{-2}) \begin{pmatrix} -\alpha_+ b_N a_N & q^{-1} b_N (\alpha_+ \mathrm{e}^\lambda + \mathrm{e}^{-\lambda}) \\ q a_N (\mathrm{e}^\lambda + \alpha_+ \mathrm{e}^{-\lambda}) & \alpha_+ a_N b_N \end{pmatrix} \right), \qquad (4.4.17)$$

$$\bar{\mathbb{B}}_1^{(\mathrm{qAL},2)} = \bar{\mathbb{B}}_1^{(\mathrm{qAL},0)} \left(\bar{\mathcal{G}}^{(\mathrm{qAL},2)} - (\mathrm{e}^{2\lambda} + \mathrm{e}^{-2\lambda}) \begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix} \right.$$

$$\left. + (1 - q^{-2}) \begin{pmatrix} -\alpha_- b_1 a_1 & b_1 (\mathrm{e}^\lambda + \alpha_- \mathrm{e}^{-\lambda}) \\ a_1 (\alpha_- \mathrm{e}^\lambda + \mathrm{e}^{-\lambda}) & \alpha_- a_1 b_1 \end{pmatrix} \right).$$

We can then use these to find the corresponding A-matrices via (4.1.20):

$$\bar{\mathbb{A}}_{N+1}^{(\mathrm{qAL},0)} = \bar{\mathbb{A}}_1^{(\mathrm{qAL},0)} = -2\mathrm{i}\ln(q) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{\mathbb{A}}_{N+1}^{(\mathrm{qAL},2)} = \mathrm{i}q^{-1}\Lambda_q - \mathrm{i}(1 - q^{-2}) \begin{pmatrix} -\alpha_+ b_N a_N & q^{-1} b_N (\alpha_+ \mathrm{e}^\lambda + \mathrm{e}^{-\lambda}) \\ q a_N (\mathrm{e}^\lambda + \alpha_+ \mathrm{e}^{-\lambda}) & \alpha_+ a_N b_N \end{pmatrix}, \quad (4.4.18)$$

$$\bar{\mathbb{A}}_1^{(\mathrm{qAL},2)} = \mathrm{i}q^{-1}\Lambda_q - \mathrm{i}(1 - q^{-2}) \begin{pmatrix} -\alpha_- b_1 a_1 & b_1 (\mathrm{e}^\lambda + \alpha_- \mathrm{e}^{-\lambda}) \\ a_1 (\alpha_- \mathrm{e}^\lambda + \mathrm{e}^{-\lambda}) & \alpha_- a_1 b_1 \end{pmatrix},$$

where we define an additional matrix Λ_q as:

$$\Lambda_q = (\mathrm{e}^{2\lambda} + \mathrm{e}^{-2\lambda}) \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}. \qquad (4.4.19)$$

Multiplying the latter two of the above A-matrices by q, as we did for the open

Hamiltonian, (4.4.14), we find the boundary A-matrices for the qAL model:

$$\bar{A}_{N+1}^{(\mathrm{qAL})} = \mathrm{i}\Lambda_q - \mathrm{i}(q - q^{-1}) \begin{pmatrix} -\alpha_+ b_N a_N & q^{-1} b_N(\alpha_+ \mathrm{e}^\lambda + \mathrm{e}^{-\lambda}) \\ q a_N(\mathrm{e}^\lambda + \alpha_+ \mathrm{e}^{-\lambda}) & \alpha_+ a_N b_N \end{pmatrix},$$

$$(4.4.20)$$

$$\bar{A}_1^{(\mathrm{qAL})} = \mathrm{i}\Lambda_q - \mathrm{i}(q - q^{-1}) \begin{pmatrix} -\alpha_- b_1 a_1 & b_1(\mathrm{e}^\lambda + \alpha_- \mathrm{e}^{-\lambda}) \\ a_1(\alpha_- \mathrm{e}^\lambda + \mathrm{e}^{-\lambda}) & \alpha_- a_1 b_1 \end{pmatrix}.$$

Using these matrices in the zero-curvature condition gives the same boundary equations as the Hamiltonian did, (4.4.15) and (4.4.16).

Chapter 5

Conclusions

The primary goal of this thesis has been to emphasise the connection between the various integrable hierarchies that can be constructed through the Lax/zero-curvature representation. In the continuous case, we can build the usual classical hierarchy with equal-time Hamiltonians, as done in Sections 2.1-2.3, or we can build the dual hierarchy with equal-space "Hamiltonians", as done in Chapter 3. In the discrete setting, we have walked through both the classical hierarchy in Sections 2.4-2.6, as well as the quantum analogue of this, that is, a hierarchy of quantum spin chains sharing a common Lax matrix, which was done in Chapter 4.

In each of these settings we have systematically constructed both the conserved quantities (conserved with respect to space evolution in the dual description of the continuous case, and time evolution otherwise) and the missing component of the Lax pair in both the cases of closed (periodic) and open (reflective) boundary conditions. The parallels between the various settings are made more evident by working with equivalent models as examples for each of these.

Following Chapter 2 (which consisted of known results and methodology, see e.g. [9], described here to contrast the later chapters against), we first focussed on continuous models, and how they can be described in terms of their space-evolution. This idea has been applied to the non-linear Schrödinger (NLS) model (in scalar [12], vector [39], and matrix [11] forms), the NLS hierarchy [13], the isotropic Landau-Lifshitz (HM) model [10], and the sine-Gordon model [40], and leads to the introduction of spatially conserved quantities and equal-space Poisson brackets, which make use of

the equal-space Hamiltonians to describe the space-evolution. The core underlying object here is the classical r-matrix, which, along with some seed Lax matrix, allows a hierarchy of Lax pairs to be extracted by generating the missing components. Then, by switching which component of the Lax pair we use as the seed matrix, we can use the same r-matrix to build a hierarchy of Lax pairs with a common V-matrix (assuming we started with a seed U-matrix originally). As the r-matrix is shared amongst all of these Poisson structures and choices of Lax matrices (as discussed in Appendix A), we view this as the fundamental object[1].

This means that starting from a Lax matrix (chosen here to be U) and an r-matrix, we generate a hierarchy of V-matrices. Then, combining each of those V-matrices with the r-matrix, we generate new hierarchies of U-matrices, which can each be used to generate their own tower of V-matrices, and so on. Diagrammatically, with $A \implies B - C - ...$ denoting that A is used with the r-matrix to generate the tower of Lax matrices containing B and C (the r-matrices are not written down, due to being shared across the whole diagram):

$$
\begin{array}{llll}
U \implies V_0 \text{———————} & V_1 \text{———————} & V_2 \text{———————} & \cdots \\
\quad\quad \Downarrow & \quad \Downarrow & \quad \Downarrow \\
\quad U_{00} \implies V_{000} - \cdots & U_{10} \implies V_{100} - \cdots & U_{20} \implies V_{200} - \cdots \\
\quad\quad | \quad\quad \Downarrow & \quad | \quad\quad \Downarrow & \quad | \quad\quad \Downarrow \\
\quad\quad | \quad\quad \vdots & \quad | \quad\quad \vdots & \quad | \quad\quad \vdots \\
\quad U_{01} \implies V_{010} - \cdots & U_{11} \implies V_{110} - \cdots & U_{21} \implies V_{210} - \cdots \\
\quad\quad | \quad\quad \Downarrow & \quad | \quad\quad \Downarrow & \quad | \quad\quad \Downarrow \\
\quad\quad | \quad\quad \vdots & \quad | \quad\quad \vdots & \quad | \quad\quad \vdots \\
\quad\quad \vdots & \quad \vdots & \quad \vdots
\end{array}
$$

$$(5.1.1)$$

---

[1] Due to the Lax matrix and r-matrix being related through the Poisson bracket relations, (2.1.8) or (3.1.4), it can suffice to only have one of the two. As a given choice of Lax matrix will usually have a unique (up to suitable transformations) r-matrix, and that the Lax matrix is directly connected to the Hamiltonian (through the hierarchy of conserved quantities) and the equations of motion (through the other component of the Lax pair), it may seem more natural to call the Lax matrix the fundamental object.

In the sense that using only the Lax matrix as the sole input datum, we can derive the r-matrix and with it the full hierarchies of both equal-space and equal-time, it is indeed more fundamental. However, if we move on to consider systems higher up in the equal-space–equal-time "lattice" then this original Lax matrix is swiftly forgotten as we work with ever more complicated Lax pairs, whereas the r-matrix remains ever present. We therefore choose to view the r-matrix as the common core of this whole construction.

The equal-time and equal-space hierarchies do not *a priori* commute (e.g. the Lax pair (U_{00}, V_{000}) is not necessarily equivalent to the pair (U_{10}, V_1) in the diagram above, despite both of these points being reached by going right once and down once from the point labelled V_0), so the diagram above keeps the "branches" separate. However, in the case where the two hierarchies do commute (which has been shown to be true for both the NLS and HM models at the lower orders of their hierarchies [10]), the diagram can be drawn more compactly:

$$
\begin{array}{c|ccccccc}
j & & & & & & & \\
& \vdots & & \vdots & & \vdots & & \\
& | & & | & & | & & \\
2 & (U_0, V_2) & \!\!\!\!-\!\!\!- & (U_1, V_2) & \!\!\!\!-\!\!\!- & (U_2, V_2) & \!\!\!\!-\!\!\!- & \ldots \\
& | & & | & & | & & \\
1 & (U_0, V_1) & \!\!\!\!-\!\!\!- & (U_1, V_1) & \!\!\!\!-\!\!\!- & (U_2, V_1) & \!\!\!\!-\!\!\!- & \ldots \\
& | & & | & & | & & \\
0 & (U_0, V_0) & \!\!\!\!-\!\!\!- & (U_1, V_0) & \!\!\!\!-\!\!\!- & (U_2, V_0) & \!\!\!\!-\!\!\!- & \ldots \\
\hline
& 0 & & 1 & & 2 & & i
\end{array}
\tag{5.1.2}
$$

Each node in this diagram represents a particular set of equations, defined in terms of the Lax pair (U_i, V_j). For example, if the HM U-matrix is used as U_0 and V_0, then the equations of motion (1.1.2) lie at the point $(i, j) = (0, 1)$ (as the HM V-matrix appears as the second term in the expansion of the Semenov-Tian-Shansky formula, (2.2.20)), and the higher system discussed in Section 3.3 is at the point $(i, j) = (2, 1)$.

This raises the evident question of whether it can be shown that for any system these equal-time and equal-space constructions do commute in general, or what the conditions of the non-commutativity are. An important consideration in this is whether the problem is being discussed at the level of the equations of motion or at the level of the Lax pair, as the Lax pair for a given system of equations is not unique. A detailed analysis of this problem is left as an open question.

The higher order systems discussed in Section 3.3 look to present a plethora of new and potentially interesting integrable models. Of particular interest are the

connections between these models and the known models that they are built from. One key difference between the models is the number of fields it contains, for example the higher order models given in Section 3.3 are both written in terms of twice as many fields as their underlying equations had (i.e. the higher NLS equations were written in terms of ψ, $\bar{\psi}$, ϕ, and $\bar{\phi}$, whereas NLS is only written using ψ and $\bar{\psi}$). By explicitly writing in the dependence on the different space-flows, these can instead be written using the same fields, but with an additional spatial parameter (i.e. as (2+1)-dimensional equations rather than (1+1)-dimensional equations):

$$\psi_y = -\psi_{xt} - 2\psi(\bar{\psi}\psi_x - \psi\bar{\psi}_x),$$

$$\bar{\psi}_y = \bar{\psi}_{xt} + 2\bar{\psi}(\bar{\psi}\psi_x - \psi\bar{\psi}_x),$$

$$\psi_{xy} = \psi_{tt} - 2\psi_t|\psi|^2 - 2\psi_x(\bar{\psi}\psi_x - \psi\bar{\psi}_x), \tag{5.1.3}$$

$$\bar{\psi}_{xy} = \bar{\psi}_{tt} + 2\bar{\psi}_t|\psi|^2 + 2\bar{\psi}_x(\bar{\psi}\psi_x - \psi\bar{\psi}_x),$$

where we use x to denote the original NLS space-flow and y to denote the new order 4 space flow. Alternatively, because the lowest order V-matrices for both the NLS and HM models coincide with their U-matrices, we can interchange the roles of the space and time coordinates in the equations of motion (by reflecting our position in (5.1.2) about the $i = j$ line) to get more traditional time-evolution equations:

$$\dot{\psi} = -\phi' - 2\psi(\bar{\psi}\phi - \psi\bar{\phi}),$$

$$\dot{\bar{\psi}} = \bar{\phi}' + 2\bar{\psi}(\bar{\psi}\phi - \psi\bar{\phi}),$$

$$\dot{\phi} = \psi'' - 2\psi'|\psi|^2 - 2\phi(\bar{\psi}\phi - \psi\bar{\phi}), \tag{5.1.4}$$

$$\dot{\bar{\phi}} = \bar{\psi}'' + 2\bar{\psi}'|\psi|^2 + 2\bar{\phi}(\bar{\psi}\phi - \psi\bar{\phi}).$$

Because these systems lie in the hierarchy of integrable models with known solitonic solutions, they should also accept solitonic solutions of a similar form, and a study of the solutions of these higher equations, as well as the relationship any such solutions would have with the original NLS and HM equations, would be a fruitful avenue of investigation.

In Chapter 3 we also studied the introduction of integrable reflective boundary conditions to the time axis for both the NLS and HM models. While seemingly unphysical, such boundary conditions could have applications as a particular type of initial condition for the system, where the time coordinate is considered on the half-line, $[0, \infty)$, instead. Thus, the boundary conditions discussed in Chapter 3 would appear as a particular set of initial conditions that settle, for example, into a 2-soliton solution.

Having both spatial and temporal boundary conditions, an interesting study would be the question of how these interact at the corners of the space-time region considered. By taking the $t \to \pm\tau$ limit of (2.3.17) and the $x \to \pm L$ limit of (3.4.13), for example, the compatibility of these boundary conditions tells us that $\psi' \propto \bar{\psi}'$ at each of the four corners. The proportionality constant is given in terms of the constants β_+ and γ_- from the temporal boundary conditions, (3.4.13). A full analysis of this, however, is left for future consideration.

The other key result of this thesis is the derivation of an analogue of the discrete classical Semenov-Tian-Shansky formula, (2.5.14), for quantum spin chains in the case of both periodic, (4.1.16), and reflective, (4.3.15), boundary conditions. This was used to generate the appropriate hierarchies of A-matrices for both the XXX Heisenberg spin chain and the quantum Ablowitz-Ladik model, which show the agreement between our formula and the expected results (at the level of the evolution equations).

Due to their general construction, the derived formulae can be applied to any quantum spin chain which accepts an R-matrix description. It is left for future work, however, to extract the hierarchy of A-matrices for other systems.

The Lax pair A-matrices have previously been a primarily classical concern, so their introduction at the quantum level will allow some of the classical techniques to be applied in this quantum setting. One example of such would be Darboux-Bäcklund transformations, as introduced in this picture in [14], which are used classically to generate solutions to PDEs, or as a description of discrete time evo-

lution. Quantum Darboux-Bäcklund transformations have been considered before, see for example [41, 42], but in previous works only their spatial components have been discussed, due to the lack of the temporal component. The consequences and application of such techniques is left for future work.

The construction of the auxiliary linear problem at the quantum level is connected to the concept of a quantum analogue of the Gelfand-Levitan-Marchenko (GLM) equation [9, 43]. This in turn leads to the potential for some quantum Zakharov-Shabat style dressing procedure [44, 45]. A non-commutative GLM equation has been constructed in [46] in the context of the classical matrix NLS model.

All of the models discussed in this thesis have been ultra-local, that is, the Poisson brackets (or commutators) have depended only on $\delta(x - y)$ (or δ_{nm}). Consequently, it would be of great interest to see if the constructions in this thesis, the equal-space hierarchy and quantum auxiliary linear problem, would be applicable to non-ultra-local models, such as the real Korteweg-de Vries system (either as a continuous model or a quantum lattice [47]), where the Poisson brackets/commutators depend on $\delta'(x - y)$ or $\delta_{n,m+1}$.

This would require the introduction of some Maillet term [48] to the algebraic relations providing the Poisson brackets or commutators, (2.1.8), (2.4.5), and (4.1.6). For dual systems, it remains an open question as to whether, if the dual construction can be applied to non-ultra-local models, the dual construction would retain the non-ultra-locality.

Appendix A

The Poisson Hierarchy for the Anisotropic Landau-Lifshitz Model

In this appendix we change our focus to the anisotropic Landau-Lifshitz (LL) equation [49, 17] in place of either the non-linear Schrödinger (NLS) or isotropic Landau-Lifshitz (HM) models of Chapter 3 which this appendix branches out from. The reason for doing this is so that we may develop a more general result, and can then focus this down to compare against the Poisson structures in Chapter 2 and the dual Poisson structures in Chapter 3. Consequently, Section A.1 is dedicated to briefly introducing this model and how it relates to the NLS and HM models.

Having introduced the relevant quantities, we turn to the problem at hand in Section A.2, where we derive a general expression for the Poisson brackets found for any given system in not just the standard hierarchy, but also for any system that can be found by alternating the equal-space and equal-time hierarchies. Section A.3 then connects this to the results from Chapters 2 and 3 and makes some closing comments.

A.1   The Anisotropic Landau-Lifshitz Model

The LL model is the set of three equations [49]:

$$\dot{\vec{S}} = \frac{\mathrm{i}}{c^2}\vec{S} \times \vec{S}'' + \mathrm{i}\vec{S} \times (J\vec{S}), \tag{A.1.1}$$

where $\vec{S}(x,t) = (S_x, S_y, S_z)^T$ is a vector containing the three fields and the 3×3 matrix $J = \mathrm{diag}(J_x, J_y, J_z)$ contains the parameters that determine the anisotropy, with $J_x < J_y < J_z$. Throughout this appendix, we will use an i, j, or k subscript to denote an element of $\{x, y, z\}$ (e.g. S_i can be any of S_x, S_y, or S_z). These anisotropy parameters appear in the Lax pair and r-matrix in the form:

$$\rho = \frac{1}{2}\sqrt{J_z - J_x}, \qquad k = \sqrt{\frac{J_y - J_x}{J_z - J_x}}, \qquad (\text{A.1.2})$$

which, due to the ordering of the J_i are required to obey $\rho > 0$ and $0 < k < 1$.

The Lax pair and r-matrix for this model are written in terms of the Jacobi elliptic functions $\mathrm{cs}(\lambda, k)$, $\mathrm{ds}(\lambda, k)$, and $\mathrm{ns}(\lambda, k)$ (see [50] for useful properties and identities). There are Lax pairs with simpler dependences on λ, e.g. linear [51], however these involve matrices larger than 2×2, so we choose to work with the elliptic versions.

We will drop the dependence on k throughout to keep things compact, and the λ parameter will also be omitted when it can be inferred from context. The Lax pair for this system can then be written [17, 52]:

$$U(\lambda) = i\rho \begin{pmatrix} S_z\mathrm{cs} & S_x\mathrm{ns} - iS_y\mathrm{ds} \\ S_x\mathrm{ns} + iS_y\mathrm{ds} & -S_z\mathrm{cs} \end{pmatrix},$$

$$V(\lambda) = 2\rho^2 \begin{pmatrix} S_z\mathrm{ds\ ns} & S_x\mathrm{cs\ ds} - iS_y\mathrm{cs\ ns} \\ S_x\mathrm{cs\ ds} + iS_y\mathrm{cs\ ns} & -S_z\mathrm{ds\ ns} \end{pmatrix} \qquad (\text{A.1.3})$$

$$+ \rho \begin{pmatrix} L_z\mathrm{cs} & L_x\mathrm{ns} - iL_y\mathrm{ds} \\ L_x\mathrm{ns} + iL_y\mathrm{ds} & -L_z\mathrm{cs} \end{pmatrix},$$

where:

$$L_i = \frac{1}{c^2} \sum_{j,k\in\{x,y,z\}} S_j' S_k \epsilon_{ijk}, \qquad (\text{A.1.4})$$

with ϵ_{ijk} being the antisymmetric Levi-Civita tensor, normalised so that $\epsilon_{xyz} = +1$.

The associated r-matrix is:

$$r(\lambda) = \frac{\mathrm{i}\eta\rho}{2} \begin{pmatrix} \mathrm{cs} & 0 & 0 & \mathrm{ns} - \mathrm{ds} \\ 0 & -\mathrm{cs} & \mathrm{ns} + \mathrm{ds} & 0 \\ 0 & \mathrm{ns} + \mathrm{ds} & -\mathrm{cs} & 0 \\ \mathrm{ns} - \mathrm{ds} & 0 & 0 & \mathrm{cs} \end{pmatrix}. \tag{A.1.5}$$

Combining this with the U-matrix in the equal-time linear algebraic relation, (2.1.8), gives an equal-time Poisson bracket of $\mathfrak{su}_2$ type:

$$\{S_i(x_1), S_j(x_2)\}_S = -\mathrm{i}\eta S_k \epsilon_{ijk}\, \delta(x_1 - x_2), \tag{A.1.6}$$

while combining the r-matrix with the V-matrix in the equal-space version, (3.1.4), gives a Poisson structure of special Euclidean SE(3) type (i.e. describing translations S_i and rotations L_i in 3 dimensions):

$$\{L_i(t_1), L_j(t_2)\}_T = L_k \epsilon_{ijk}\, \delta(t_1 - t_2),$$

$$\{L_i(t_1), S_j(t_2)\}_T = S_k \epsilon_{ijk}\, \delta(t_1 - t_2), \tag{A.1.7}$$

$$\{S_i(t_1), S_j(t_2)\}_T = 0,$$

These Poisson structures are equivalent to those for the isotropic version presented in Chapters 2 and 3.

A.1.1 Limits

In order to connect this with the results in Chapters 2 and 3, we provide here the limits that take this model to the NLS and HM models [17].

Isotropic Landau-Lifshitz

As the name might imply, the HM model can be found by considering the isotropic limit of the LL model, that is, where $J_x \to J_y \to J_z$ (or $\rho, k \to 0$), so that the anisotropic component of the equations of motion, (A.1.1), vanishes. We also set $\eta = 1$ and $\lambda \to 2\mathrm{i}\lambda\rho$ for the conventions to match up.

As this limit leaves the fields unchanged, the standard and dual Poisson structures above are preserved in the limit.

Non-linear Schrödinger

To extract the NLS model from the LL model, we need a more delicate limit than for the HM model, where we allow a partial anisotropy, $J_y \to J_x, J_z = J_x - \frac{1}{\eta}$ (or $\rho = \frac{i}{2\sqrt{\eta}}$ and $k \to 0$). Then, c is set to $c = 1$ and the fields S_i are replaced with combinations of the NLS fields, ψ and $\bar{\psi}$ through:

$$S_x = \sqrt{\eta} \left(e^{-t/\eta}\psi + e^{t/\eta}\bar{\psi} \right), \qquad S_y = -i\sqrt{\eta} \left(e^{-t/\eta}\psi - e^{t/\eta}\bar{\psi} \right),$$

$$S_z = \sqrt{1 - 4\eta|\psi|^2}.$$

(A.1.8)

Finally, the limit $\eta \to 0$ is taken.

A.2 The Hierarchy of Poisson Structures

The goal of this appendix is to establish that each system of equations in the hierarchy of integrable equations associated to the anisotropic Landau-Lifshitz model admits a Poisson structure. Specifically that the Semenov-Tian-Shansky formula [6], (2.2.20), can be applied to each of the V-matrices (along with (A.1.5) as the r-matrix) generated from the U-matrix, and similarly for each of the U-matrices generated by the V-matrices (and so on). Thus, not only will each of the systems in the standard equal-time hierarchy admit such, but in fact every system of equations that can be extracted from alternating the equal-space and equal-time pictures will admit Poisson structures too.

Throughout the following calculations we focus on the standard equal-time picture (i.e. generating V-matrices from U-matrices), which means that all of the derived Poisson brackets will be dual Poisson brackets in the sense of Chapter 3. The calculations and results are identical however for the dual (equal-space) picture, so we merely quote the equivalent result at the end.

The first step in this is to determine the form of any Lax matrices generated by the Semenov-Tian-Shansky formula (2.2.20) (or (3.2.18) for the dual case).

A.2.1   The Form of the Lax Matrix

The V-matrices are generated from the U- and r-matrices for the system by (2.2.20):

$$\mathbb{V}_b = \mathsf{t}^{-1}(\mu)\, \mathrm{tr}_a \left\{ T_a(L, x; \mu) r_{ab}(\mu - \lambda) T_a(x, -L; \mu) \right\}. \tag{A.2.1}$$

This was handled in Section 2.2 by decomposing the monodromy matrices T into a combination of W- and Z-matrices. Then, as the W-matrices are wholly anti-diagonal, where we label its two non-zero components as w_{12} and w_{21}, after splitting the monodromy matrices with the diagonalisation (2.2.5), the V-matrix generator can be written as:

$$\mathbb{V}_b = \frac{1}{1 - w_{12}(\mu) w_{21}(\mu)} \mathrm{tr}_a \left\{ r_{ab}(\mu - \lambda) \begin{pmatrix} 1 & -w_{12}(\mu) \\ w_{21}(\mu) & -w_{12}(\mu) w_{21}(\mu) \end{pmatrix}_a \right\}. \tag{A.2.2}$$

Explicitly evaluating this, we get:

$$\mathbb{V} = \frac{\mathrm{i}\eta\rho}{2(1 - w_{12} w_{21})} \begin{pmatrix} (1 + w_{12} w_{21})\mathrm{cs} & (w_{21} - w_{12})\mathrm{ns} - (w_{21} + w_{12})\mathrm{ds} \\ (w_{21} - w_{12})\mathrm{ns} + (w_{21} + w_{12})\mathrm{ds} & -(1 + w_{12} w_{21})\mathrm{cs} \end{pmatrix}. \tag{A.2.3}$$

Thus, the generator of the V-matrices (or U-matrices in the dual case) can be written in the form:

$$\mathbb{V}(\lambda, \mu) = \begin{pmatrix} \mathbb{V}_z(\lambda, \mu) & \mathbb{V}_x(\lambda, \mu) - \mathrm{i}\mathbb{V}_y(\lambda, \mu) \\ \mathbb{V}_x(\lambda, \mu) + \mathrm{i}\mathbb{V}_y(\lambda, \mu) & -\mathbb{V}_z(\lambda, \mu) \end{pmatrix}. \tag{A.2.4}$$

Due to the association of cs with the z-component, ns with the x-component, and ds with the y-component, we define $\mathrm{ps}_x = \mathrm{ns}$, $\mathrm{ps}_y = \mathrm{ds}$, and $\mathrm{ps}_z = \mathrm{cs}$ so that the results can be combined into one.

A.2.2   The Linear Algebraic Relation

As we want to find the Poisson structure for each of these V-matrices, we need to consider the dual algebraic relation (3.1.4) (we suppress the time dependence and trailing Dirac delta function to keep the expression compact):

$$\{\mathbb{V}_a(\lambda_1, \mu), \mathbb{V}_b(\lambda_2, \mu)\}_T = [r_{ab}(\lambda_1 - \lambda_2), \mathbb{V}_a(\lambda_1, \mu) + \mathbb{V}_b(\lambda_2, \mu)]. \tag{A.2.5}$$

Evaluating this using (A.1.5) and (A.2.4), we get:

$$\{\mathbb{V}_i(\lambda_1, \mu), \mathbb{V}_j(\lambda_2, \mu)\}_T = \eta\rho\big(\mathrm{ps}_i(\lambda_1 - \lambda_2)\mathbb{V}_k(\lambda_2, \mu) - \mathrm{ps}_j(\lambda_1 - \lambda_2)\mathbb{V}_k(\lambda_1, \mu)\big)\epsilon_{ijk}.$$

$$(\text{A.2.6})$$

In reference to (A.2.3), we can separate the dependence on λ in each of these by defining:

$$\mathbb{V}_i(\lambda, \mu) = -\eta\rho\mathbb{S}_i(\mu)\mathrm{ps}_i(\mu - \lambda). \qquad (\text{A.2.7})$$

It will be useful in what follows to compact the elliptic functions by writing $\mathrm{ps}_{i,jk} = \mathrm{ps}_i(\lambda_j - \lambda_k)$ with $\lambda_0 = \mu$. Then, as we want to focus on the system at a specific order in the hierarchy (labelled n), the Poisson brackets for the generating $\mathbb{V}_i$ in (A.2.6) become:

$$0 = \left\{\frac{1}{n!}\partial_\mu^n\big(\mathrm{ps}_{i,01}\mathbb{S}_i(\mu)\big), \frac{1}{n!}\partial_\mu^n\big(\mathrm{ps}_{j,02}\mathbb{S}_j(\mu)\big)\right\}_T\Big|_{\mu=0}$$

$$+ \left(\frac{1}{n!}\mathrm{ps}_{i,12}\partial_\mu^n\big(\mathrm{ps}_{k,02}\mathbb{S}_k(\mu)\big) - \frac{1}{n!}\mathrm{ps}_{j,12}\partial_\mu^n\big(\mathrm{ps}_{k,01}\mathbb{S}_k(\mu)\big)\right)\Big|_{\mu=0}\epsilon_{ijk},$$

$$(\text{A.2.8})$$

where we have used derivatives with respect to μ to pick out the term in the hierarchy that we are interested in. After evaluating the derivatives, with a bracketed superscript $^{(n)}$ denoting the nth term in the power series expansion with respect to μ, this becomes:

$$0 = \sum_{p,q=0}^{n}\frac{1}{n!}\binom{n}{p}\binom{n}{q}\mathrm{ps}_{i,01}^{(n-p)}\mathrm{ps}_{j,02}^{(n-q)}\{\mathbb{S}_i^{(p)}, \mathbb{S}_j^{(q)}\}_T$$

$$(\text{A.2.9})$$

$$+ \sum_{p=0}^{n}\binom{n}{p}(\mathrm{ps}_{i,12}\mathrm{ps}_{k,02}^{(n-p)} - \mathrm{ps}_{j,12}\mathrm{ps}_{k,01}^{(n-p)})\mathbb{S}_k^{(p)}\epsilon_{ijk}.$$

The elliptic functions can be shown to satisfy the following identity:

$$0 = \mathrm{ns}(a - b)\mathrm{ds}(a - c) + \mathrm{cs}(c - a)\mathrm{ds}(c - b) + \mathrm{cs}(b - a)\mathrm{ns}(b - c), \qquad (\text{A.2.10})$$

which, if we differentiate $(n - p)$ times with respect to $\lambda_0 = \mu$ and convert to our

notation (with i, j, k distinct) gives us:

$$\text{ps}_{i,12}\text{ps}_{k,02}^{(n-p)} - \text{ps}_{j,12}\text{ps}_{k,01}^{(n-p)} = -\sum_{q=0}^{n-p}\binom{n-p}{q}\text{ps}_{j,01}^{(n-p-q)}\text{ps}_{i,02}^{(q)}. \tag{A.2.11}$$

We can use this in (A.2.9) to make every term depend on $\text{ps}_{i,01}$ and $\text{ps}_{j,02}$ and no other elliptic functions:

$$0 = \sum_{p,q=0}^{n} \frac{1}{n!}\binom{n}{p}\binom{n}{q}\text{ps}_{i,01}^{(n-p)}\text{ps}_{j,02}^{(n-q)}\{\mathbb{S}_i^{(p)}, \mathbb{S}_j^{(q)}\}_T$$
$$-\sum_{q=0}^{n}\sum_{p=n-q}^{n}\binom{n}{p+q-n}\binom{2n-q-p}{n-q}\text{ps}_{j,01}^{(n-p)}\text{ps}_{i,02}^{(n-q)}\mathbb{S}_k^{(p+q-n)}\epsilon_{ijk}. \tag{A.2.12}$$

We can therefore split these expressions about their elliptic function dependence to get two sets of results (where n is the order of the system you are considering and p and q lie in $[0, n]$ and supply you with the $(n+1)^2$ Poisson brackets between the $(n+1)$ fields):

$$\{\mathbb{S}_i^{(p)}, \mathbb{S}_j^{(q)}\}_T = \begin{cases} \frac{p!q!}{(p+q-n)!}\mathbb{S}_k^{(p+q-n)}\epsilon_{ijk} & p+q \geq n, \\ 0 & p+q < n. \end{cases} \tag{A.2.13}$$

In order to state this more cleanly, we redefine the fields one last time, where instead of the components $\mathbb{S}_i^{(p)}$ we define the fields in the system at order n as:

$$S_i^{(n,p)} = \frac{1}{(n-p)!}\mathbb{S}_i^{(n-p)}, \tag{A.2.14}$$

so that the Poisson brackets in (A.2.13) become:

$$\{S_i^{(n,p)}(t_1), S_j^{(n,q)}(t_2)\}_T = \begin{cases} S_k^{(n,p+q)}\epsilon_{ijk}\,\delta(t_1 - t_2) & p+q \leq n, \\ 0 & p+q > n. \end{cases} \tag{A.2.15}$$

Thus, each Poisson structure in the hierarchy of such for the anisotropic Landau-Lifshitz model has the form of a graded Lie algebra with an $\mathfrak{su}_2$-like form.

In the dual picture where we use the underlying V-matrix to generate a hierarchy of U-matrices and then wish to find the equal-time brackets for each of the corresponding systems, the above calculation proceeds exactly the same, with the

equivalent of (A.2.15) being:

$$\{S_i^{(n,p)}(x_1), S_j^{(n,q)}(x_2)\}_S = \begin{cases} S_k^{(n,p+q)} \epsilon_{ijk}\, \delta(x_1 - x_2) & p+q \le n, \\ 0 & p+q > n. \end{cases}$$

(A.2.16)

In the discussion below we drop the $_S$ or $_T$ subscript from the bracket, as the results are equivalent in the two cases.

A.3 Comments and Examples

In analogy to how the standard and dual algebras in (A.1.6) and (A.1.7) have one and two Casimir elements respectively ($c^2 = \sum_i S_i^2$ for (A.1.6) and both c^2 and $\tilde{c}^2 = \sum_i L_i S_i$ for (A.1.7)), we can find $(n+1)$ such constants for the Poisson algebra at order n in the hierarchy (indexed by $0 \le r \le n$):

$$c_{n,r} = \sum_i \sum_{p=0}^{r} S_i^{(n,n-p)} S_i^{(n,n+p-r)}.$$

(A.3.1)

By inspection we can see that this coincides with our known Casimir elements as $c^2 = c_{n,n}$ and $\tilde{c}^2 = \frac{1}{2}c_{n,n-1}$, where we are recognising $S_i = S_i^{(n,n)}$ and $L_i = S_i^{(n,n-1)}$. To see that this is a Casimir we consider the Poisson bracket of this with $S_j^{(q)}$ (we drop the factors of δ and the n superscripts to clean the expressions a little bit):

$$
\begin{aligned}
\{c_{n,r}, S_j^{(q)}\} &= \sum_i \sum_{p=0}^{r} \{S_i^{(n-p)} S_i^{(n+p-r)}, S_j^{(q)}\} \\
&= \sum_{i,k} \left(\sum_{p=0}^{r-q} S_i^{(n-p)} S_k^{(n+p+q-r)} + \sum_{p=q}^{r} S_i^{(n+p-r)} S_k^{(n+q-p)} \right) \epsilon_{ijk} \\
&= \sum_{i,k} \sum_{p=q}^{r} (S_i^{(n+q-p)} S_k^{(n+p-r)} + S_i^{(n+p-r)} S_k^{(n+q-p)}) \epsilon_{ijk} \\
&= 0,
\end{aligned}
$$

(A.3.2)

where in the final step we have used the antisymmetry of the Levi-Civita tensor.

Despite the results of this appendix depending only on the choice of n to determine the form of the Poisson structure, this result holds for any choice of underlying Lax pair. This is because the only tool used in the derivation of the results was the r-matrix. Consequently, as long as the underlying Lax matrix satisfies the appropriate linear algebraic relation, (2.1.8), with this choice of r-matrix, then each dual system in its hierarchy will also satisfy a linear algebraic relation with the same r-matrix, with the choice of n in (A.2.15) depending on the order at which the generated half of the Lax pair appears at the hierarchy.

For example, if we take the NLS limit (described below in Subsection A.3.1), then we will find that the equal-space Poisson structure of the complex mKdV equation is the same as the equal-time Poisson structure of the $(4,3)$-NLS equations described in Section 3.3. This is because the complex mKdV V-matrix (out of which the equal-space brackets are built) appears at the same power of the spectral parameter as the U-matrix (which is used to build the equal-time brackets) does for the $(4,3)$-NLS model.

A.3.1 Examples

Considering when $n = 0$, (A.2.15) is simply the $\mathfrak{su}_2$ algebra, (A.1.6), which is the standard Poisson structure for both the isotropic and anisotropic Landau-Lifshitz models, while for $n = 1$ (A.2.15) becomes the SE(3) algebra, (A.1.7), which is the dual Poisson structure for both of these models. When $n = 2$, we get the following algebra (with $S_i = S_i^{(2)}$, $L_i = S_i^{(1)}$, and $T_i = S_i^{(0)}$):

$$\{T_i, T_j\} = T_k \epsilon_{ijk}, \qquad \{T_i, L_j\} = L_k \epsilon_{ijk}, \qquad \{T_i, S_j\} = S_k \epsilon_{ijk},$$

$$\{L_i, L_j\} = S_k \epsilon_{ijk}, \qquad \{L_i, S_j\} = \{S_i, S_j\} = 0. \tag{A.3.3}$$

This has three Casimir elements:

$$c_{2,0} = \sum_i S_i^2, \qquad c_{2,1} = 2 \sum_i L_i S_i, \qquad c_{2,2} = \sum_i (2T_i S_i + L_i^2), \quad \text{(A.3.4)}$$

and is (up to a suitable choice of S_i, L_i, and T_i) an equal-time Poisson structure for the higher HM system described in Section 3.3 in Chapter 3.

NLS Limit

If we assume that each of the fields $S_x^{(n,p)}$, $S_y^{(n,p)}$ can be considered of the form of (A.1.8), that is:

$$S_x^{(n,p)} = \sqrt{\eta}\left(e^{-t/\eta}\psi^{(n,p)} + e^{t/\eta}\bar{\psi}^{(n,p)}\right),$$

$$S_y^{(n,p)} = -i\sqrt{\eta}\left(e^{-t/\eta}\psi^{(n,p)} - e^{t/\eta}\bar{\psi}^{(n,p)}\right),$$

(A.3.5)

then the appropriate Casimir elements at order n can be used to find the corresponding limit for $S_z^{(n,r)}$. For $r = n$, this is equivalent to the limit in (A.1.8):

$$S_z^{(n,n)} = \sqrt{c_{n,0} - 4\eta|\psi^{(n,n)}|^2},$$

(A.3.6)

while for $r \neq n$ this leads to a recurrence relation that is close to Catalan type:

$$S_z^{(n,r)} = \frac{1}{2S_z^{(n,n)}}\left(a_{n,r} - \sum_{p=1}^{n-r-1} S_z^{(n,n-p)}S_z^{(n,r+p)}\right),$$

(A.3.7)

where:

$$a_{n,r} = c_{n,n-r} - \sum_{p=0}^{n-r}\left(S_x^{(n,n-p)}S_x^{(n,r+p)} + S_y^{(n,n-p)}S_y^{(n,r+p)}\right).$$

(A.3.8)

In the original NLS limit we chose $c = 1$ (i.e. $c_{n,0} = 1$), which we do again here. However, we also need to choose $c_{n,p} = 0$ for all $0 < p \leq n$, in the same way that we worked with $\tilde{c} = 0$ in the isotropic case. Then, in the limit as $\eta \to 0$ these $a_{n,r}$ become:

$$a_{n,r} = -4\eta\sum_{p=0}^{n-r}\psi^{(n,n-p)}\bar{\psi}^{(n,r+p)}.$$

(A.3.9)

The recurrence relation (A.3.7) can be solved to get a solution:

$$S_z^{(n,r)} = -\sum_{m=1}^{n-r}\frac{(2m-2)!}{m!(m-1)!}\left(2S_z^{(n,n)}\right)^{1-2m}\sum_{\substack{i_1,\ldots,i_m=1\\\sum_j i_j=n-r}} a_{n,n-i_1}\ldots a_{n,n-i_m},$$

(A.3.10)

however, as we only care about the order η^0 and η^1 terms and the $a_{n,r}$ are linear in η, there are no order η^0 contributions from $S_z^{(n,r)}$ with $r < n$ and the order η^1 contribution is:

$$S_z^{(n,r)} = \delta_{rn} - 2\eta\sum_{p=0}^{n-r}\psi^{(n,n-p)}\bar{\psi}^{(n,r+p)} + \ldots.$$

(A.3.11)

The resulting hierarchy of NLS Poisson structures then all take the form (noting that due to the factor of η hidden inside (A.2.7), we need to scale the right-hand side of (A.2.15) by η):

$$\{\psi^{(n,p)}, \psi^{(n,q)}\} = \{\bar{\psi}^{(n,p)}, \bar{\psi}^{(n,q)}\} = 0,$$

$$\{\bar{\psi}^{(n,p)}, \psi^{(n,q)}\} - \{\psi^{(n,p)}, \bar{\psi}^{(n,q)}\} = \mathrm{i}\delta_{p+q,n},$$

$$\{\psi^{(n,p)}, \bar{\psi}^{(n,q)}\} = \frac{-\mathrm{i}\psi^{(n,p+q)}}{2\psi^{(n,n)}} - \sum_{m=1}^{n-q} \frac{\psi^{(n,n-m)}}{\psi^{(n,n)}}\{\psi^{(n,p)}, \bar{\psi}^{(n,q+m)}\}, \qquad \text{(A.3.12)}$$

$$\{\bar{\psi}^{(n,p)}, \psi^{(n,q)}\} = \frac{\mathrm{i}\bar{\psi}^{(n,p+q)}}{2\bar{\psi}^{(n,n)}} - \sum_{m=1}^{n-q} \frac{\bar{\psi}^{(n,n-m)}}{\bar{\psi}^{(n,n)}}\{\bar{\psi}^{(n,p)}, \psi^{(n,q+m)}\}.$$

To actually extract the Poisson brackets from these we assume that the brackets take the form:

$$\{\psi^{(n,p)}, \bar{\psi}^{(n,q)}\} = b_{p,q}^{(n)} - \frac{\mathrm{i}}{2}\delta_{p+q,n}, \qquad \text{(A.3.13)}$$

for some function $b_{p,q}^{(n)}$. With this assumption, the second relation tells us that $b_{q,p}^{(n)} + b_{p,q}^{(n)} = 0$. The third and fourth relations then become:

$$b_{p,q}^{(n)} = \frac{\mathrm{i}\psi^{(n,p+q)}}{2\psi^{(n,n)}}\left(\sum_{m=1}^{n-q}\delta_{p+q+m,n} - 1\right) + \frac{\mathrm{i}}{2}\delta_{p+q,n} - \sum_{m=1}^{n-q}\frac{\psi^{(n,n-m)}}{\psi^{(n,n)}}b_{p,q+m}^{(n)},$$

$$b_{p,q}^{(n)} = \frac{\mathrm{i}\bar{\psi}^{(n,p+q)}}{2\bar{\psi}^{(n,n)}}\left(1 - \sum_{m=1}^{n-q}\delta_{p+q+m,n}\right) - \frac{\mathrm{i}}{2}\delta_{p+q,n} - \sum_{m=1}^{n-q}\frac{\bar{\psi}^{(n,n-m)}}{\bar{\psi}^{(n,n)}}b_{p,q+m}^{(n)},$$

$$\text{(A.3.14)}$$

where the delta outside of the brackets can be taken into the summation, thus ensuring that the bracketed terms are identically 0. Then, equating the remaining expressions, we have:

$$0 = \sum_{m=1}^{n-q}\left(\frac{\psi^{(n,n-m)}}{\psi^{(n,n)}} - \frac{\bar{\psi}^{(n,n-m)}}{\bar{\psi}^{(n,n)}}\right)b_{p,q+m}^{(n)}. \qquad \text{(A.3.15)}$$

By considering successive orders of q (starting with $q = n - 1$) and using the fact that $\psi^{(n,p)}$ and $\bar{\psi}^{(n,q)}$ are independent, we can see that each of the $b_{p,q}^{(n)}$ must be 0. Therefore, the NLS analogue of (A.2.15), i.e. the associated hierarchy of Poisson

brackets for the system appearing at order n in the hierarchy, are:

$$\{\psi^{(n,p)}(a), \psi^{(n,q)}(b)\} = \{\bar{\psi}^{(n,p)}(a), \bar{\psi}^{(n,q)}(b)\} = 0,$$

$$\{\psi^{(n,p)}(a), \bar{\psi}^{(n,q)}(b)\} = \frac{-i}{2}\delta_{p+q,n}\delta(a-b),$$

(A.3.16)

where we generically use a and b for the parameters so that this result can be applied to both the equal-space and equal-time descriptions.

Looking at the $n = 0$ case, we extract the usual NLS Poisson brackets (2.1.9), labelling $\psi = \sqrt{2i}\psi^{(0,0)}$ and $\bar{\psi} = \sqrt{2i}\bar{\psi}^{(0,0)}$:

$$\{\psi(a), \bar{\psi}(b)\} = \delta(a-b). \tag{A.3.17}$$

If we instead consider the $n = 1$ case, labelling $\psi = \psi^{(1,0)}$, $\bar{\psi} = \bar{\psi}^{(1,0)}$, $\phi = -2i\psi^{(1,1)}$, $\bar{\phi} = 2i\bar{\psi}^{(1,1)}$, we re-derive the dual Poisson structure for NLS, (3.1.5):

$$\{\psi, \phi\} = \{\bar{\psi}, \bar{\phi}\} = \{\psi, \bar{\psi}\} = \{\phi, \bar{\phi}\} = 0,$$

$$\{\psi(a), \bar{\phi}(b)\} = \{\bar{\psi}(a), \phi(b)\} = \delta(a-b).$$

(A.3.18)

Finally, by looking at the $n = 2$ case we can find equal-space Poisson brackets for the complex mKdV equation, or equal-time brackets for the (4, 3)-NLS system defined in Section 3.3, where we use $\psi_i = \psi^{(n,i)}$ and $\bar{\psi}_i = \bar{\psi}^{(n,i)}$:

$$\{\psi_i, \psi_j\} = \{\bar{\psi}_i, \bar{\psi}_j\} = 0,$$

$$\{\psi_0, \bar{\psi}_0\} = \{\psi_0, \bar{\psi}_1\} = \{\psi_1, \bar{\psi}_0\} = \{\psi_1, \bar{\psi}_2\} = \{\psi_2, \bar{\psi}_1\} = \{\psi_2, \bar{\psi}_2\} = 0, \quad \text{(A.3.19)}$$

$$\{\psi_0(a), \bar{\psi}_2(b)\} = \{\psi_1(a), \bar{\psi}_1(b)\} = \{\psi_2(a), \bar{\psi}_0(b)\} = \frac{-i}{2}\delta(a-b).$$

To connect this equal-space Poisson structure for the complex mKdV equation to the version presented in [32], we need to transform our six fields as $\psi_0 = \frac{1}{4}b_1$, $\bar{\psi}_0 = \frac{1}{4}c_1$, $\psi_1 = \frac{1}{2\sqrt{2i}}b_2$, $\bar{\psi}_1 = \frac{1}{2\sqrt{2i}}c_2$, $\psi_2 = \frac{1}{2i}b_3 + \frac{1}{16i}b_1^2 c_1$, and $\bar{\psi}_2 = \frac{1}{2i}c_3 + \frac{1}{16i}b_1 c_1^2$.

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