Dual Integrable Models and Time-Like Boundary Conditions

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Based off of a joint work with A. Doikou and S. Sklaveniti (arXiv:1810.10937), as well as a forthcoming paper.

Talk Outline

- Lax Pairs, r-Matrices, and K-Matrices
- Constructing Hierarchies
  - Example: Non-Linear Schrödinger
  - Example: Isotropic Landau-Lifshitz
- Considering "Space-Evolution"
  - Example: Non-Linear Schrödinger
  - Example: Isotropic Landau-Lifshitz
- Combining the Pictures

Lax Pairs

The auxiliary linear problem is the pair of relations [Lax '68; Ablowitz, *et al.* '74]:

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi,$$

where the matrices (U, V) are called the Lax pair. The compatibility of these two equations is called the zero-curvature condition:

$$0 = U_t - V_x + [U, V].$$

For a given Lax pair, the associated equations of motion can be found by inserting the U- and V-matrices into this relation.

r-Matrices

Given an r-matrix that satisfies the classical Yang-Baxter equation [Semenov-Tian-Shansky '83]:

$$[r_{12}(\lambda-\mu), r_{13}(\lambda)] + [r_{12}(\lambda-\mu), r_{23}(\mu)] + [r_{13}(\lambda), r_{23}(\mu)] = 0,$$

then the Poisson brackets associated to a Lax matrix U can be found through [Sklyanin '79]:

$$\{U_1(x,\lambda), U_2(y,\mu)\}_S = [r_{12}(\lambda-\mu), U_1(x,\lambda) + U_2(y,\mu)]\, \delta(x-y).$$

For all of the models in this talk the r-matrices are proportional to:

$$\mathcal{P}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

K-Matrices

To incorporate non-periodic, non-vanishing boundary conditions into a system on the interval $[-L, L]$, we introduce some matrices $K_\pm$ that lie at the $\pm L$ boundaries.

So that the integrability of the system is preserved, we need the $K_\pm$-matrices to satisfy [Sklyanin '87]:

$$0 = K_{\pm,1}(\lambda)r_{12}(\lambda+\mu)K_{\pm,2}(\mu) - K_{\pm,2}(\mu)r_{12}(\lambda+\mu)K_{\pm,1}(\lambda)$$
$$+ [r_{12}(\lambda-\mu), K_{\pm,1}(\lambda)K_{\pm,2}(\mu)].$$

Constructing Hierarchies

Build monodromy matrix $T_S(x, y; \lambda)$ from the spatial component of the Lax pair:

$$T_S(x, y; \lambda) = P \exp \int_y^x U(s; \lambda) \mathsf{d}s.$$

Taking the trace of the full monodromy matrix $T_S(L, -L; \lambda)$ gives the transfer matrix:

$$\mathfrak{t}_S(\lambda) = \mathsf{tr}\, \{T_S(L, -L; \lambda)\}.$$

When considering open boundary conditions, the transfer matrix becomes [Sklyanin '87]:

$$\bar{\mathfrak{t}}_S(\lambda) = \mathsf{tr}\, \{K_+(\lambda) T_S(L, -L; \lambda) K_-(\lambda) T_S^{-1}(L, -L; -\lambda)\}.$$

Expanding $\mathcal{G}_S = \ln \mathfrak{t}_S$ and $\bar{\mathcal{G}}_S = \ln \bar{\mathfrak{t}}_S$ about λ:

$$\mathcal{G}_S(\lambda) = \sum_k \lambda^k \mathcal{G}_S^{(k)}, \qquad \bar{\mathcal{G}}_S(\lambda) = \sum_k \lambda^k \bar{\mathcal{G}}_S^{(k)},$$

gives a series of quantities that Poisson commute with one another (with respect to $\{\cdot,\cdot\}_S$):

$$\{\mathcal{G}_S^{(k)}, \mathcal{G}_S^{(j)}\}_S = 0 = \{\bar{\mathcal{G}}_S^{(k)}, \bar{\mathcal{G}}_S^{(j)}\}_S.$$

Treating one of these $\mathcal{G}_S$ as the Hamiltonian gives a tower of commuting conserved quantities.

For each conserved quantity $\mathcal{G}_S^{(k)}$ there is a corresponding V-matrix such that Hamilton's equation and the zero-curvature condition give the same evolution equations. These V-matrices are generated by [Semenov-Tian-Shansky '83]:

$$\mathbb{V}_2(x;\lambda,\mu) = \mathfrak{t}_S^{-1}(\mu)\mathrm{tr}_1\left\{T_{S,1}(L,x;\mu)r_{12}(\mu-\lambda)T_{S,1}(x,-L;\mu)\right\},$$

and for open boundary conditions [Avan, Doikou '08]:

$$\begin{aligned}\bar{\mathbb{V}}_2(x) = \bar{\mathfrak{t}}_S^{-1}(\mu)\,\mathrm{tr}_1\{&K_{+,1}(\mu)T_{S,1}(L,x;\mu)r_{12}^{-}T_{S,1}(x,-L;\mu)K_{-,1}(\mu)T_{S,1}^{-1}(-\mu)\\ &+K_{+,1}(\mu)T_{S,1}(\mu)K_{-,1}(\mu)T_{S,1}^{-1}(x,-L;-\mu)r_{12}^{+}T_{S,1}^{-1}(L,x;-\mu)\}.\end{aligned}$$

The boundary conditions then arise from requiring:

$$\lim_{x\to\pm L}\bar{\mathbb{V}}(x) = \bar{\mathbb{V}}(\pm L).$$

Non-Linear Schrödinger

The equations of motion are:

$$-\mathrm{i}\psi_t = \psi_{xx} - 2\kappa\psi|\psi|^2, \qquad \mathrm{i}\bar{\psi}_t = \bar{\psi}_{xx} - 2\kappa\bar{\psi}|\psi|^2.$$

These come from the Hamiltonian and Poisson brackets:

$$H_S = \int_{-L}^{L} \left(\kappa|\psi|^4 - \psi_{xx}\bar{\psi}\right)\mathrm{d}x, \qquad \{\psi(x), \bar{\psi}(y)\}_S = \mathrm{i}\,\delta(x-y),$$

which are found from the Lax pair [Zakharov, Shabat '71]:

$$U = \begin{pmatrix} \frac{\lambda}{2\mathrm{i}} & \sqrt{\kappa}\bar{\psi} \\ \sqrt{\kappa}\psi & -\frac{\lambda}{2\mathrm{i}} \end{pmatrix}, \qquad V = \mathrm{i}\sqrt{\kappa}\begin{pmatrix} \sqrt{\kappa}|\psi|^2 & -\bar{\psi}_x \\ \psi_x & -\sqrt{\kappa}|\psi|^2 \end{pmatrix} - \lambda U,$$

and the r-matrix $r_{12}(\lambda) = \frac{-\kappa}{\lambda}\mathcal{P}_{12}$.

Isotropic Landau-Lifshitz

The equations of motion are (using $\vec{S} = (S_1, S_2, S_3)^T$):

$$\vec{S}_t = \mathsf{i}\vec{S} \times \vec{S}_{xx}.$$

These come from the Hamiltonian and Poisson brackets:

$$H_S = \frac{1}{2}\int_{-L}^{L} |\vec{S}_x|^2 \mathsf{d}x, \qquad \{S_i(x), S_j(y)\}_S = -\mathsf{i}\epsilon_{ijk} S_k\, \delta(x-y).$$

These in turn are found from the Lax pair [Takhtajan '77]:

$$U = \frac{1}{2\lambda}\begin{pmatrix} S_3 & S_1 - \mathsf{i}S_2 \\ S_1 + \mathsf{i}S_2 & -S_3 \end{pmatrix} = \frac{1}{2\lambda} S, \qquad V = \frac{1}{2\lambda^2} S - \frac{1}{2\lambda} S_x S,$$

and the r-matrix $r_{12}(\lambda) = \frac{1}{2\lambda}\mathcal{P}_{12}$.

For dual models, the dual Poisson brackets associated to the time component of the Lax pair, V, can be found through [Avan, *et al.* '16]:

$$\{V_1(t_1,\lambda), V_2(t_2,\mu)\}_T = \left[\tilde{r}_{12}(\lambda-\mu), V_1(t_1,\lambda)+V_2(t_2,\mu)\right]\delta(t_1-t_2),$$

where $\tilde{r}_{12}$ is a solution of the classical Yang-Baxter equation. Then, if the dual Hamiltonian is known, the space-evolution equations are found through a dual version of Hamilton's equation:

$$\partial_x V = \{H_T, V\}_T.$$

Consider instead the dual picture, where we study space-evolution. The equal-space monodromy matrix is:

$$T_T(t_1, t_2; \lambda) = P \exp \int_{t_2}^{t_1} V(s; \lambda) \mathsf{d}s.$$

The trace of the full monodromy matrix $T_T(\tau, -\tau; \lambda)$, on the interval $[-\tau, \tau]$, gives the equal-space transfer matrix:

$$\mathfrak{t}_T(\lambda) = \mathsf{tr}\{T_T(\tau, -\tau; \lambda)\}.$$

When considering open boundary conditions, the transfer matrix becomes [Doikou, IF, Sklaveniti '18]:

$$\bar{\mathfrak{t}}_T(\lambda) = \mathsf{tr}\{K_+(\lambda) T_T(\tau, -\tau; \lambda) K_-(\lambda) T_T^{-1}(\tau, -\tau; -\lambda)\}.$$

Constructing Hierarchies

Dual Conserved Quantities

Expanding $\mathcal{G}_T = \ln \mathfrak{t}_T$ and $\bar{\mathcal{G}}_T = \ln \bar{\mathfrak{t}}_T$ about λ:

$$\mathcal{G}_T(\lambda) = \sum_k \lambda^k \mathcal{G}_T^{(k)}, \qquad \bar{\mathcal{G}}_T(\lambda) = \sum_k \lambda^k \bar{\mathcal{G}}_T^{(k)},$$

gives a series of quantities that Poisson commute with one another (with respect to $\{\cdot,\cdot\}_T$):

$$\{\mathcal{G}_T^{(k)}, \mathcal{G}_T^{(j)}\}_T = 0 = \{\bar{\mathcal{G}}_T^{(k)}, \bar{\mathcal{G}}_T^{(j)}\}_T.$$

Treating one of these $\mathcal{G}_T$ as the dual Hamiltonian gives a tower of commuting conserved (with respect to space-evolution) quantities.

For each conserved quantity $\mathcal{G}_T^{(k)}$ there is a corresponding U-matrix such that Hamilton's equation and the zero-curvature condition give the same evolution equations. These U-matrices are generated by [Avan, *et al.* '16]:

$$\mathbb{U}_2(t;\lambda,\mu) = \mathfrak{t}_T^{-1}(\mu)\mathsf{tr}_1\left\{T_{T,1}(\tau,t;\mu)\tilde{r}_{12}(\mu-\lambda)T_{T,1}(t,-\tau;\mu)\right\},$$

while the in the case of open boundary conditions it is [Doikou, IF, Sklaveniti '18]:

$$\begin{aligned}\bar{\mathbb{U}}_2(t) = \bar{\mathfrak{t}}_T^{-1}(\mu)\,\mathsf{tr}_1\{&K_{+,1}(\mu)T_{T,1}(\tau,t;\mu)\tilde{r}^-_{12}T_{T,1}(t,-\tau;\mu)K_{-,1}(\mu)T^{-1}_{T,1}(-\mu)\\ &+K_{+,1}(\mu)T_{T,1}(\mu)K_{-,1}(\mu)T^{-1}_{T,1}(t,-\tau;-\mu)\tilde{r}^+_{12}T^{-1}_{T,1}(\tau,t;-\mu)\}.\end{aligned}$$

The boundary conditions then arise from requiring:

$$\lim_{t\to\pm\tau}\bar{\mathbb{U}}(t) = \bar{\mathbb{U}}(\pm\tau).$$

The dual equations of motion are [Avan, *et al.* '16]:

$$\psi_x = \phi, \qquad \bar{\psi}_x = \bar{\phi},$$
$$\phi_x = 2\kappa\psi|\psi|^2 - \mathrm{i}\psi_t, \qquad \bar{\phi}_x = 2\kappa\bar{\psi}|\psi|^2 + \mathrm{i}\bar{\psi}_t.$$

These are generated by the dual Hamiltonian:

$$H_T = \int_{-\tau}^{\tau} \left(\kappa|\psi|^4 - |\phi|^2 + \mathrm{i}\psi\bar{\psi}_t\right)\mathrm{d}t,$$

via the dual Poisson brackets (found from $\tilde{r}_{12}(\lambda) = -r_{12}(\lambda) = \frac{\kappa}{\lambda}\mathcal{P}_{12}$):

$$\{\psi(t_1), \bar{\phi}(t_2)\}_T = \{\bar{\psi}(t_1), \phi(t_2)\}_T = \delta(t_1 - t_2),$$

with the rest being trivial.

The most general boundary K-matrix is [de Vega, González-Ruiz '94]:

$$K_{\pm} = \frac{a_{\pm}}{\lambda}\mathbb{I} + \begin{pmatrix} d_{\pm} & b_{\pm} \\ c_{\pm} & -d_{\pm} \end{pmatrix}.$$

The boundary conditions at $t = \pm\tau$ are:

$$\begin{aligned} t &= +\tau: \qquad & d_{+} &= 0, \qquad & \bar{\psi} &= a_{+} + b_{+}\psi, \\ t &= -\tau: \qquad & d_{-} &= 0, \qquad & \psi &= a_{-} + c_{-}\bar{\psi}. \end{aligned}$$

For comparison, the boundary conditions at $x = \pm L$ are:

$$b_{\pm} = c_{\pm} = 0, \qquad d_{\pm} = -1,$$
$$\psi_x = \pm a_{\pm}\psi, \qquad \bar{\psi}_x = \pm a_{\pm}\bar{\psi}.$$

The dual equations of motion are (using $\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)^T$):

$$\vec{S}_x = \vec{\Sigma}, \qquad \vec{\Sigma}_x = \mathrm{i}\vec{S} \times \vec{S}_t - \vec{S}|\vec{\Sigma}|^2.$$

These are found from the dual Hamiltonian:

$$H_T = \frac{1}{2}\int_{-\tau}^{\tau} \left(|\vec{\Sigma}|^2 - 2\mathrm{i}\frac{S_{1,t}S_2 - S_1 S_{2,t}}{1 + S_3} \right) \mathrm{d}t,$$

via the dual Poisson brackets (found by using $\tilde{r}_{12}(\lambda) = r_{12}(\lambda) = \frac{1}{2\lambda}\mathcal{P}_{12}$), which have two Casimir elements, $|\vec{S}|^2$ and $\vec{S} \cdot \vec{\Sigma}$:

$$\{S_i(t_1), S_j(t_2)\}_T = 0,$$
$$\{S_i(t_1), \Sigma_j(t_2)\}_T = \left(S_i S_j - \delta_{ij}\right) \delta(t_1 - t_2),$$
$$\{\Sigma_i(t_1), \Sigma_j(t_2)\}_T = \left(S_i \Sigma_j - S_j \Sigma_i\right) \delta(t_1 - t_2).$$

The boundary K-matrix is chosen to be equivalent to the one for NLS:

$$K_\pm = a_\pm \mathbb{I} + \lambda \begin{pmatrix} d_\pm & \beta_\pm - \mathrm{i}\gamma_\pm \\ \beta_\pm + \mathrm{i}\gamma_\pm & -d_\pm \end{pmatrix}.$$

At $t = \pm\tau$ the boundary conditions are:

$$a_\pm = 0, \qquad\qquad 0 = \beta_\pm S_1 + \gamma_\pm S_2 + d_\pm S_3.$$

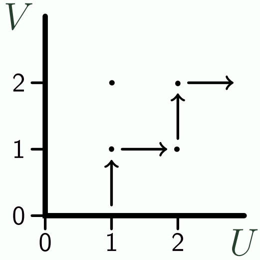
For comparison, the boundary conditions at $x = \pm L$ are:

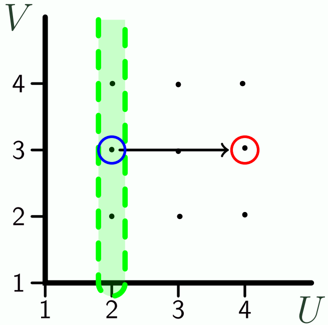
$$\begin{aligned} a_\pm(S_{1,x}S_2 - S_1 S_{2,x}) &= \pm(\beta_\pm S_2 - \gamma_\pm S_1), \\ a_\pm(S_1 S_{3,x} - S_{1,x}S_3) &= \pm(d_\pm S_1 - \beta_\pm S_3), \\ a_\pm(S_2 S_{3,x} - S_{2,x}S_3) &= \pm(d_\pm S_2 - \gamma_\pm S_3). \end{aligned}$$

Combining Hierarchies

The standard picture gives a hierarchy of integrable systems (defined by their time-evolution), while the dual picture gives a hierarchy of integrable systems (defined by their space-evolution).

By alternating which picture is considered, we can build a 2-dimensional "lattice" of integrable systems. This is not *a priori* commutative.





Blue circle - The NLS model

Green dashed region - The usual NLS hierarchy

Red circle - A new, 4-field system

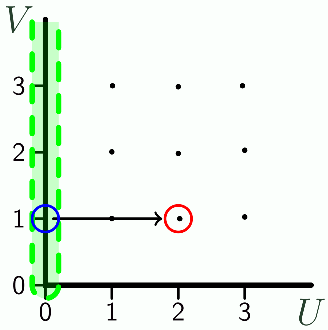
The usual NLS is at (2, 3). We can consider instead (4, 3). The Lax pair for this model re-uses the V-matrix from the standard NLS model:

$$U = \sqrt{\kappa}\begin{pmatrix} \sqrt{\kappa}(\psi\bar{\phi} - \phi\bar{\psi}) & -\mathrm{i}\bar{\psi}_t \\ \mathrm{i}\psi_t & -\sqrt{\kappa}(\psi\bar{\phi} - \phi\bar{\psi}) \end{pmatrix} - \lambda V,$$
$$V = \begin{pmatrix} -\frac{\lambda^2}{2\mathrm{i}} + \mathrm{i}\kappa|\psi|^2 & -\sqrt{\kappa}(\lambda\bar{\psi} + \mathrm{i}\bar{\psi}_x) \\ -\sqrt{\kappa}(\lambda\psi - \mathrm{i}\psi_x) & \frac{\lambda^2}{2\mathrm{i}} - \mathrm{i}\kappa|\psi|^2 \end{pmatrix}.$$

The corresponding equations of motion are:

$$\psi_x = 2\kappa\psi(\bar{\psi}\phi - \psi\bar{\phi}) + \mathrm{i}\phi_t,$$
$$\bar{\psi}_x = 2\kappa\bar{\psi}(\psi\bar{\phi} - \bar{\psi}\phi) - \mathrm{i}\bar{\phi}_t,$$
$$\phi_x = 2\kappa\phi(\bar{\psi}\phi - \psi\bar{\phi}) + 2\mathrm{i}\kappa|\psi|^2\psi_t + \psi_{tt},$$
$$\bar{\phi}_x = 2\kappa\bar{\phi}(\psi\bar{\phi} - \bar{\psi}\phi) - 2\mathrm{i}\kappa|\psi|^2\bar{\psi}_t + \bar{\psi}_{tt}.$$

Isotropic Landau-Lifshitz



Blue circle - The isotropic LL model

Green dashed region - The usual isotropic LL hierarchy

Red circle - A new, 6-field system

The usual ILL is at (0, 1). We can consider instead (2, 1). The Lax pair for this model re-uses the V-matrix from the standard ILL model:

$$U = \frac{1}{2\lambda^3}S - \frac{1}{2\lambda^2}\Sigma S - \frac{1}{2\lambda}\dot{S}S + \frac{1}{4\lambda}\Sigma^2 S,$$
$$V = \frac{1}{2\lambda^2}S - \frac{1}{2\lambda}\Sigma S.$$

The corresponding equations of motion are:

$$\vec{S}_x = \mathsf{i}(\vec{S} \times \vec{\Sigma}_t) + \frac{1}{2}|\vec{\Sigma}|^2\vec{\Sigma},$$
$$\vec{\Sigma}_x = \mathsf{i}(\vec{\Sigma} \times \vec{\Sigma}_t) - \frac{\mathsf{i}}{2}|\vec{\Sigma}|^2(\vec{S} \times \vec{S}_t) + \mathsf{i}\vec{\Sigma}\big(\vec{\Sigma} \cdot (\vec{S} \times \vec{S}_t)\big)$$
$$+ \vec{S}_{tt} + \vec{S}(|\vec{S}_t|^2 - \frac{1}{2}|\vec{\Sigma}|^4).$$

Future Work

- Building a dual construction of the fully anisotropic Landau-Lifshitz (both NLS and the isotropic LL are special cases of this).
- Finding a solution to an equation with time-like boundary conditions (via, e.g., inverse scattering on the half-line).
- Performing a more detailed investigation of the properties of the higher order systems (e.g. looking for physical interpretations, finding solutions).